OLIVER, WAIT, AND JONES' MATHAMATICS.

A TREATISE ON PROJECTIVE GEOMETRY.

BY

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CONTENTS.

PROJECTIVE GEOMETRY.

☞ In this book all figures are plane figures, unless otherwise stated; and by a line is meant a straight line.

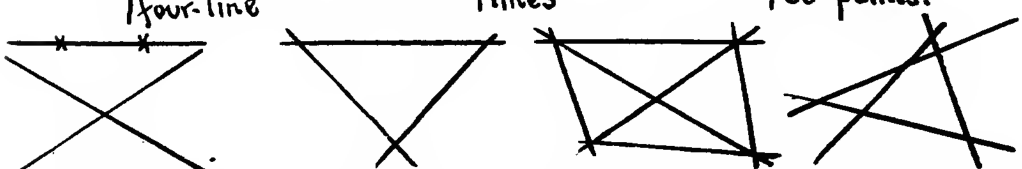
§ 1. DUALITY.

In a plane there may be two kinds of figures:

point-figures, made up of points and the lines that join them, their *co-lines*.

line-figures, made up of lines and the points where they meet, their *co-points*.

e.g., a	two-point two-line	consists of two	points lines	and their single	co-line. co-point.
a	three-point three-line	consists of three	points lines	and their three	co-lines. co-points.
a	four-point four-line	consists of four	points lines	and their six	co-lines. co-points.



A point-figure and a line-figure are *correlative* if the first have as many points as the other has lines, as in the examples above.

A point and a line are *correlative elements*. In projective geometry the properties of point-figures and of line-figures may be set forth in *correlative propositions*, i.e. propositions that relate to point-figures and to their correlative line-figures in the same way, and

which may in general, be got one from the other by interchanging the words point and line. A pair of correlative propositions form a *dual proposition*.

$$\text{e.g. two}\left\{\begin{array}{l}\text{points}\\ \text{lines}\end{array}\right.\text{give a single}\left\{\begin{array}{l}\text{line}\\ \text{point}\end{array}\right.;\quad n\left\{\begin{array}{l}\text{points}\\ \text{lines}\end{array}\right.\text{give }\tfrac{1}{2}n(n-1)\left\{\begin{array}{l}\text{lines.}\\ \text{points.}\end{array}\right.$$

In space points and planes are correlative elements and so are lines and lines. A point-figure is correlative to a plane-figure, and a line-figure to a line-figure.

$$\text{e.g., two}\left\{\begin{array}{l}\text{points}\\ \text{planes}\end{array}\right.\text{give a line; three}\left\{\begin{array}{l}\text{points}\\ \text{planes}\end{array}\right.\text{give a}\left\{\begin{array}{l}\text{plane.}\\ \text{point.}\end{array}\right.$$

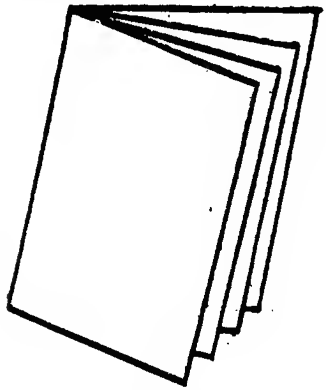
$$\text{so, a}\left\{\begin{array}{l}\text{point}\\ \text{plane}\end{array}\right.\text{and a line give a}\left\{\begin{array}{l}\text{plane.}\\ \text{point.}\end{array}\right.$$

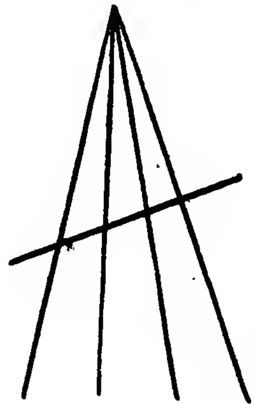
§ 2. PERSPECTIVE.

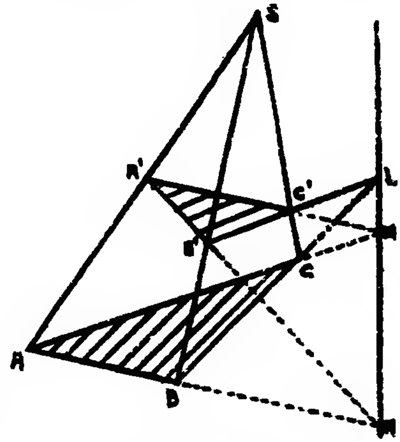
If a figure consist of lines thro a fixed point, or of surfaces (planes and cones) generated by such lines, such a figure is a *sheaf*. The fixed point is the *centre*, and the lines are *rays*.

If a sheaf be cut by two planes, *transversals*, the sections are *figures in perspective*, and each of them is an *image* of the other. The co-line of the two transversals is the *axis*. The lines and surfaces that form the sheaf *project* the two figures into each other, point into point, line into line, and curve into curve.

A sheaf of lines that lie in a common plane is a *pencil*. A transversal cuts the rays of a pencil in a *range* of points. A sheaf of planes that meet in a common line is a *book* and the planes of a book are its *leaves*. A transversal cuts the leaves of a book in a pencil of rays.







THEOR. 1. *If two figures be in perspective, the image of a {range / pencil} is a {range / pencil}.*

For the projector of a {range / pencil} is a {pencil / book}, that is cut by any transversal in a {range. / pencil.}

Q.E.D.

THEOR. 2. *If two figures be in perspective a line and its image meet upon the axis.* [Geom.

§3. HOMOLOGY.

If two figures in perspective be projected from the same centre upon the same plane, the two co-planar figures so formed are in *homology*, and each is the image of the other. The figure above, §2, if regarded as a pyramid in space and two plane sections of it, shows two triangles in perspective; but the actual drawing as made upon the paper is co-planar and shows two triangles in homology. It is the image with the eye as centre, of the pyramid and its sections.

THEOR. 3. *If two figures be in homology, the image of a* $\left\{\begin{array}{l}\text{range}\\\text{pencil}\end{array}\right.$ *is a* $\left\{\begin{array}{l}\text{range.}\\\text{pencil.}\end{array}\right.$

[th. 1. df. hom.

THEOR. 4. *If two figures be in homology, a* $\left\{\begin{array}{l}\text{point}\\\text{line}\end{array}\right.$ *and its image* $\left\{\begin{array}{l}\text{project}\\\text{reflect}\end{array}\right.$ *into each other from a fixed* $\left\{\begin{array}{l}\text{centre.}\\\text{axis}\end{array}\right.$

For ∵ the two $\left\{\begin{array}{l}\text{points}\\\text{lines}\end{array}\right.$ that are images of each other in the primary perspective $\left\{\begin{array}{l}\text{lie on a line thro}\\\text{meet in a point of}\end{array}\right.$ the $\left\{\begin{array}{l}\text{centre}\\\text{axis}\end{array}\right.$ of perspective.

[Df. persp., th. 2

∴ their images in the plane of homology $\left\{\begin{array}{l}\text{lie on a line thro}\\\text{meet in a point of}\end{array}\right.$ the image of the $\left\{\begin{array}{l}\text{centre}\\\text{axis}\end{array}\right.$ of the perspective.

[th. 1

Q.E.D.

A $\left\{\begin{array}{l}\text{point}\\\text{line}\end{array}\right.$ that is its own image is a *double* $\left\{\begin{array}{l}\text{point.}\\\text{line.}\end{array}\right.$

THEOR. 5. *If two figures be in homology, the* $\left\{\begin{array}{l}\text{centre}\\\text{axis}\end{array}\right.$ *and every* $\left\{\begin{array}{l}\text{point of}\\\text{line thro}\end{array}\right.$ *the* $\left\{\begin{array}{l}\text{axis}\\\text{centre}\end{array}\right.$ *are double* $\left\{\begin{array}{l}\text{points;}\\\text{lines;}\end{array}\right.$ *and there are no other double* $\left\{\begin{array}{l}\text{points}\\\text{lines}\end{array}\right.$ *unless the figures coincide throughout.*

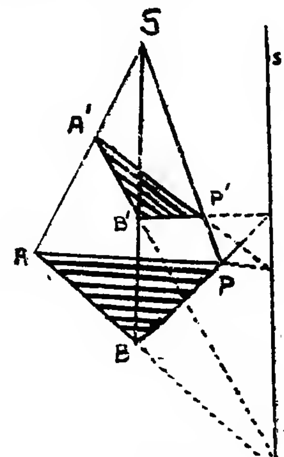
For the $\left\{\begin{array}{l}\text{line}\\\text{plane}\end{array}\right.$ that projects the $\left\{\begin{array}{l}\text{centre,}\\\text{axis,}\end{array}\right.$ or any $\left\{\begin{array}{l}\text{point on the axis}\\\text{line thro the centre}\end{array}\right.$ of the primary perspective, projects the two primary images $\left\{\begin{array}{l}\text{on}\\\text{in}\end{array}\right.$ $\begin{array}{l}\text{that line}\\\text{that plane}\end{array}$ into coincident $\left\{\begin{array}{l}\text{points}\\\text{lines}\end{array}\right.$ of the homology.

And no other $\left\{\begin{array}{l}\text{line}\\ \text{plane}\end{array}\right.$ can project two primary images into coincident images unless the centre of projection coincide with the centre of perspective; and then the two figures in homology coincide throughout.

COR. *If two figures be in homology, and if there be four double* $\left\{\begin{array}{l}\textit{points,}\\ \textit{lines,}\end{array}\right.$ *no three of which are* $\left\{\begin{array}{l}\textit{colinear,}\\ \textit{concurrent,}\end{array}\right.$ *the two figures coincide throughout.*

<u>THEOR</u>. 6. *If in two flat point-figures* AB...P..., A'B'...P'... *the lines* AB, AP, BP, *reflect into the lines* A'B', A'P', B'P' *from an axis* S, *for every pair of points* P, P', *the two figures are in perspective or homology.*

(a) *the two figures not co-planar.*



For ∵ AB, A'B' meet,
∴ they are coplanar.
So AP, A'P' are coplanar, BP, B'P' are coplanar.
∴ the co-lines AA', BB', PP' of these three planes meet in S, the co-point of AA', BB'
∴ the two figures are in perspective.

Q.E.D.

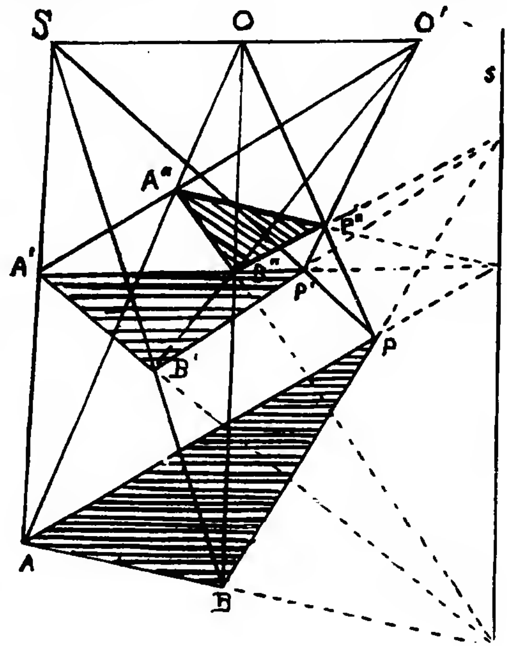
(b) *the two figures co-planar.*

For, in any plane through s oblique to the plane of the given figures, draw a point-figure A"B"... P"... whose co-lines A"B", A"P", B"P", meet the like lines of AB...P..., A'B'...P'... on s;
then ∵ AB...P..., A"B"...P"... are in perspective from some centre O, [(a)

and $AB\ldots P\ldots, A'B'\ldots P'\ldots$ are their co-planar projections from some centre O', [(a)

$\therefore AB\ldots P\ldots, A'B'\ldots P'\ldots$ are in homology. [df. hom.

Q.E.D.



COR. *If two point-figures $AB\ldots P\ldots$, $A'B'\ldots P'\ldots$ be in perspective or in homology, and either figure be turned about the axis, they are always in perspective or in homology, the same points are images to the same points, and the same lines to the same lines.*

Homology is therefore a limiting case of perspective, and figures in homology may hereafter be included under the title of figures in perspective.

THEOR. 7. *If in two flat line-figures $ab\ldots p\ldots$, $a'b'\ldots p'\ldots$ the points ab, ap, bp, project into the points $a'b'$, $a'p'$, $b'p'$, from a centre S, for every pair of lines p, p', the two figures are in perspective.*

(a) *the two figures not co-planar:* [df. persp.

(b) *the two figures co-planar:*

For take O, O' any centres in space colinear with S, and let a'', b'', p'' be the co-lines of the projecting planes Oa, $O'a'$, Ob, $O'b'$, Op, $O'p'$; then $\because$ O, O' are colinear with S, and so are the points ab, $a'b'$ [hyp.

∴ the lines O-ab, O'-a'b' are co-planar and meet;

and ∵ their co-point lies in the co-line of O-a, O'-a' and in that of O-b, O'-b',

∴ it is common to a'', b'';

i.e. a'', b'' meet and are co-planar.

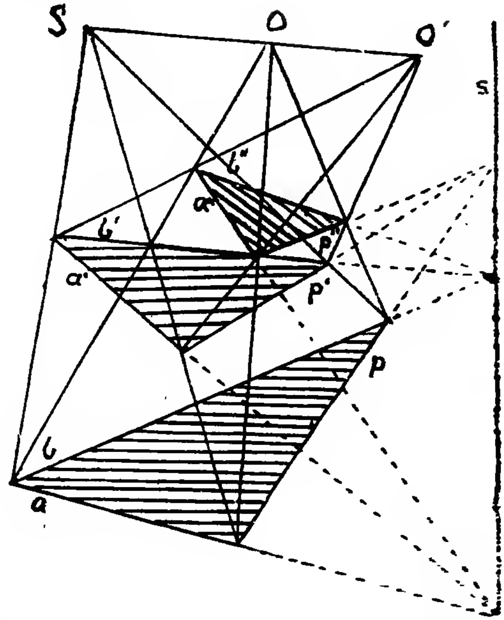
So O-ap, O'a'p' meet and their co-point is common to a'', p'';

i.e. p'' meets a''.

So O-bp, O'-b'p' meet and their co-point is common to b'', p'';

i.e. p'' meets b''.

∴ a'', b'', p'' are co-planar.

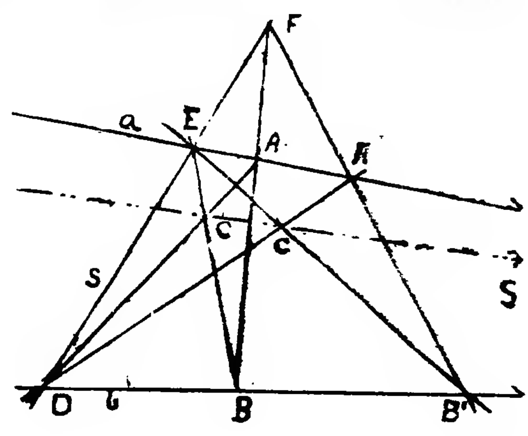


And ∵ ab...p..., a''b''...p''... are in perspective from O, and project from O' into the co-planar figures ab...p..., a'b'...p'..., [constr

∴ ab...p..., a'b'...p' are in homology. Q E D

PROB. 1. *To draw the* $\left\{\begin{matrix}\text{co-line}\\ \text{co-point}\end{matrix}\right.$ *of a given* $\left\{\begin{matrix}\text{point}\\ \text{line}\end{matrix}\right.$ *and the unknown* $\left\{\begin{matrix}\text{co-point}\\ \text{co-line}\end{matrix}\right.$ *of two given* $\left\{\begin{matrix}\text{lines:}\\ \text{points:}\end{matrix}\right.$

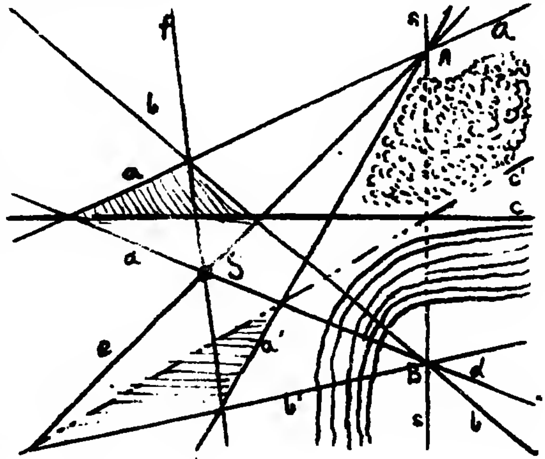
Let $\left\{\begin{matrix}C, a, b\\ c, A, B\end{matrix}\right.$ be the given $\left\{\begin{matrix}\text{point}\\ \text{line}\end{matrix}\right.$ and $\left\{\begin{matrix}\text{lines; on } a, b\\ \text{points; thro } A, B\end{matrix}\right.$ take any $\left\{\begin{matrix}\text{points } A, B\\ \text{lines } a, b\end{matrix}\right.$ and let $\left\{\begin{matrix}D \equiv CA\text{-}b,\ E \equiv BC\text{-}a,\ F \equiv AB\text{-}DE,\\ d \equiv ca\text{-}B,\ e \equiv bc\text{-}A,\ f \equiv ab\text{-}de,\end{matrix}\right.$ and $\left\{\begin{matrix}\text{thro } F\\ \text{on } f\end{matrix}\right.$ take any $\left\{\begin{matrix}\text{line}\\ \text{point}\end{matrix}\right.$ that



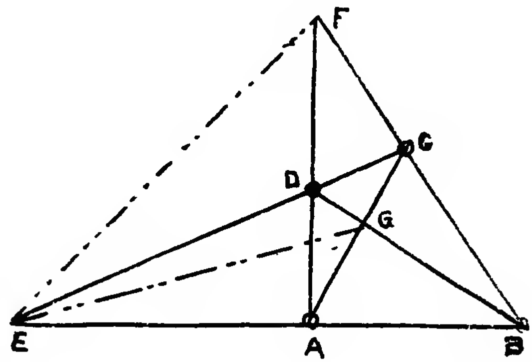
{meets a, b in A′, B′ / joins to A, B by a′, b′} and let {C ≡ A′D–B′E; / c ≡ a′d–b′e;} then ∵ the two {three-points ABC, A′B′C′ / three-lines abc, a′b′c′} are in perspective with {axis DEF / centre def.} [th. 6 / th. 7]

∴ The {line CC′ is concurrent / point cc′ is colinear} with {a, b. / A, B.}

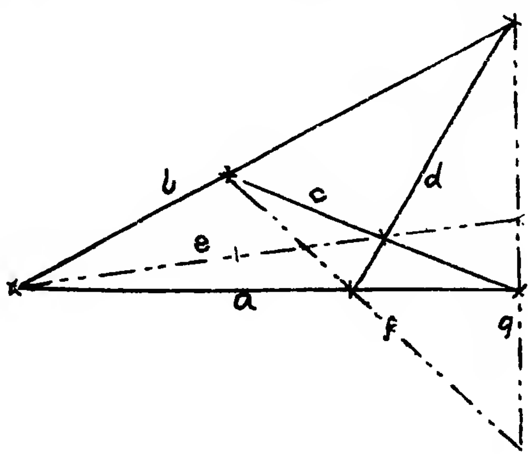
Q.E.F.



The four {points / lines} that form a {four-point / four-line} are its {vertices, / sides,} and their six {co-lines / co-points} are its {sides. / vertices.}



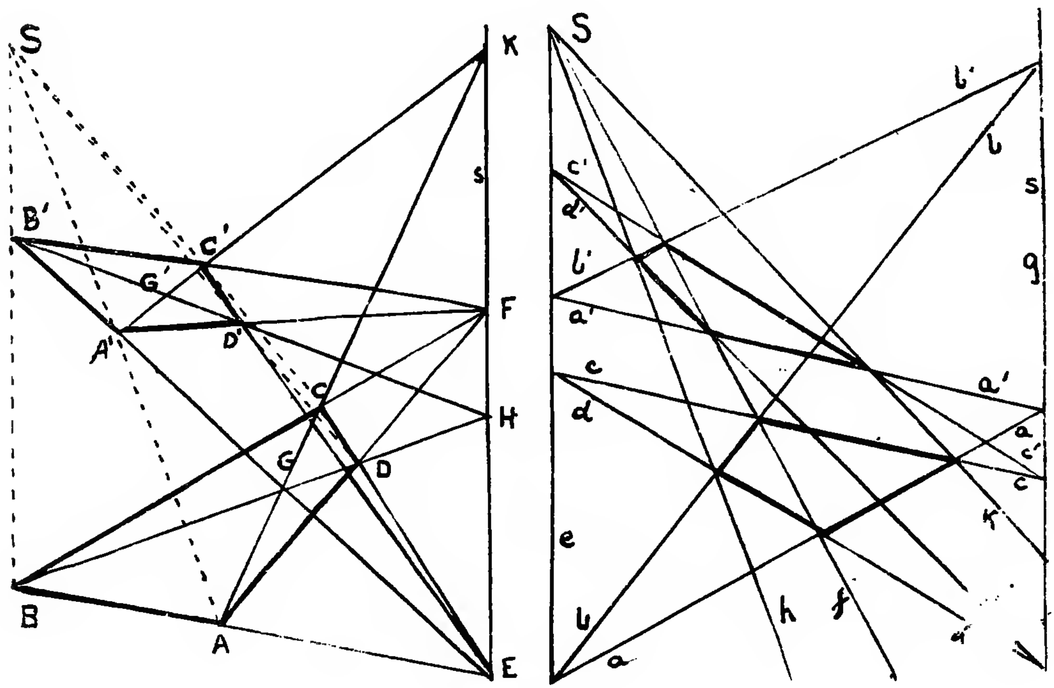
Two {sides / vertices} whose {co-point / co-line} is not a {vertex / side} are *opposite* {*sides* / *vertices*} and their {co-point / co-line} is a *diagonal* {*point.* / *line.*} The three diagonal {points / lines} of a {four-point / four-line} form its *diagonal triangle.*



In the figure {A, B, C, D / a, b, c, d} are the

$\begin{cases}\text{vertices}\\\text{sides}\end{cases}$ of the $\begin{cases}\text{four-point,}\\\text{four-line,}\end{cases}$ and $\begin{cases}AB, CD, & BC, AD, & CA, BD\\ab, cd, & bc, ad, & ca, bd\end{cases}$ are its three pairs of opposite $\begin{cases}\text{sides;} & E, F, G.\\\text{vertices;} & e, f, g.\end{cases}$ are the three diagonal $\begin{cases}\text{points,}\\\text{lines,}\end{cases}$ and $\begin{cases}EFG.\\efg\end{cases}$ is the diagonal triangle.

THEOR. 8. *If in a variable* $\begin{cases}\textit{four-point}\\\textit{four-line}\end{cases}$ *the diagonal* $\begin{cases}\textit{points}\\\textit{lines}\end{cases}$ *given by two pairs of opposite* $\begin{cases}\textit{sides}\\\textit{vertices}\end{cases}$ *be fixed and a fifth* $\begin{cases}\textit{side pass thro}\\\textit{vertex lie on}\end{cases}$ *a fixed* $\begin{cases}\textit{point on}\\\textit{line thro}\end{cases}$ *the* $\begin{cases}\textit{co-line}\\\textit{co-point}\end{cases}$ *of the two diagonal* $\begin{cases}\textit{points,}\\\textit{lines,}\end{cases}$ *the sixth* $\begin{cases}\textit{side passes thro}\\\textit{vertex lies on}\end{cases}$ *a fixed* $\begin{cases}\textit{point on}\\\textit{line thro}\end{cases}$ *this* $\begin{cases}\textit{co-line.}\\\textit{co-point.}\end{cases}$ $\begin{bmatrix}\text{th. 6.}\\\text{th. 7.}\end{bmatrix}$



EXAMPLES.

1\. Given centre S, axis s, a $\left\{\begin{matrix}\text{point-figure } ABC\ldots\\ \text{line-figure } abc\ldots\end{matrix}\right.$ and a $\left\{\begin{matrix}\text{point } A' \text{ colinear}\\ \text{line } a' \text{ concurrent}\end{matrix}\right.$ with $\left\{\begin{matrix}S, A\\ s, a\end{matrix}\right.$ to construct the image $\left\{\begin{matrix}A'B'C'\ldots\\ a'b'c'\ldots\end{matrix}\right.$ of $\left\{\begin{matrix}ABC\ldots.\\ abc\ldots.\end{matrix}\right.$

2\. Given $\left\{\begin{matrix}\text{axis } s, \text{ a point-figure } ABC\ldots, \text{ a point } A', \text{ a point } B' \text{ colinear}\\ \text{centre } S, \text{ a line-figure } abc\ldots, \text{ a line } a', \text{ a line } b' \text{ concurrent}\end{matrix}\right.$ $\begin{matrix}\text{with } A' \text{ and } s\text{-}AB\\ \text{with } a' \text{ and } S\text{-}ab\end{matrix}$ to find the $\left\{\begin{matrix}\text{centre } S\\ \text{axis } s\end{matrix}\right.$ and to construct the image of $\left\{\begin{matrix}ABC\ldots.\\ abc\ldots.\end{matrix}\right.$

3\. Given $\left\{\begin{matrix}\text{centre } S, \text{ a point-figure } ABC\ldots\\ \text{axis } s, \text{ a line-figure } abc\ldots\end{matrix}\right.$ and three $\left\{\begin{matrix}\text{points } A', B', C',\\ \text{lines } a', b', c',\end{matrix}\right.$ $\begin{matrix}\text{colinear}\\ \text{concurrent}\end{matrix}$ with $\left\{\begin{matrix}S \text{ and } A, B, C,\\ s \text{ and } a, b, c,\end{matrix}\right.$ to find $\left\{\begin{matrix}\text{axis } s\\ \text{centre } S\end{matrix}\right.$ and to construct the image of $\left\{\begin{matrix}ABC\ldots.\\ abc\ldots.\end{matrix}\right.$

4\. In ex. 1, 2, 3 let the given figure be a circle.

5\. Through a given point to draw a line parallel to two given parallel lines.

6\. Through a given point to draw a line parallel to a given line, by aid of a given parallelogram.

From the centre of the parallelogram as centre and with the opposite sides of it as images of each other, construct the images of the co-points of the parallelogram and the given line; the problem is then reduced to ex. 5.

7\. Through a given point to draw a line perpendicular to a given line, by aid of a given circle and its centre.

Draw two diameters of the circle and form a parallelogram by joining their extremities; then draw a parallel to the given line as in Ex. 6 cutting the circle, and draw a perpendicular to it by the principle that an angle inscribed in a half circle is a right angle.

omit

8. If the three $\left\{\begin{matrix}\text{vertices}\\ \text{sides}\end{matrix}\right.$ of a variable triangle $\left\{\begin{matrix}\text{slide}\\ \text{turn}\end{matrix}\right.$ on three $\left\{\begin{matrix}\text{rays}\\ \text{points}\end{matrix}\right.$ of a fixed $\left\{\begin{matrix}\text{pencil}\\ \text{range}\end{matrix}\right.$ while two of its $\left\{\begin{matrix}\text{sides turn}\\ \text{vertices slide}\end{matrix}\right.$ on two $\left\{\begin{matrix}\text{points}\\ \text{rays}\end{matrix}\right.$ of a fixed $\left\{\begin{matrix}\text{range,}\\ \text{pencil,}\end{matrix}\right.$ then will the third $\left\{\begin{matrix}\text{side turn}\\ \text{vertex slide}\end{matrix}\right.$ on a third $\left\{\begin{matrix}\text{point}\\ \text{ray}\end{matrix}\right.$ of that $\left\{\begin{matrix}\text{range.}\\ \text{pencil.}\end{matrix}\right.$

omit

9. If a line turn on a fixed point S and cut a pair of lines a, a' in P, P' the locus of the co-point of rays OP, O'P', drawn from any centres O, O' that are colinear with $\left\{\begin{matrix}\text{S}\\ \text{aa'}\end{matrix}\right.$, is a line through the point aa',

§4. VANISHING POINTS AND LINES.

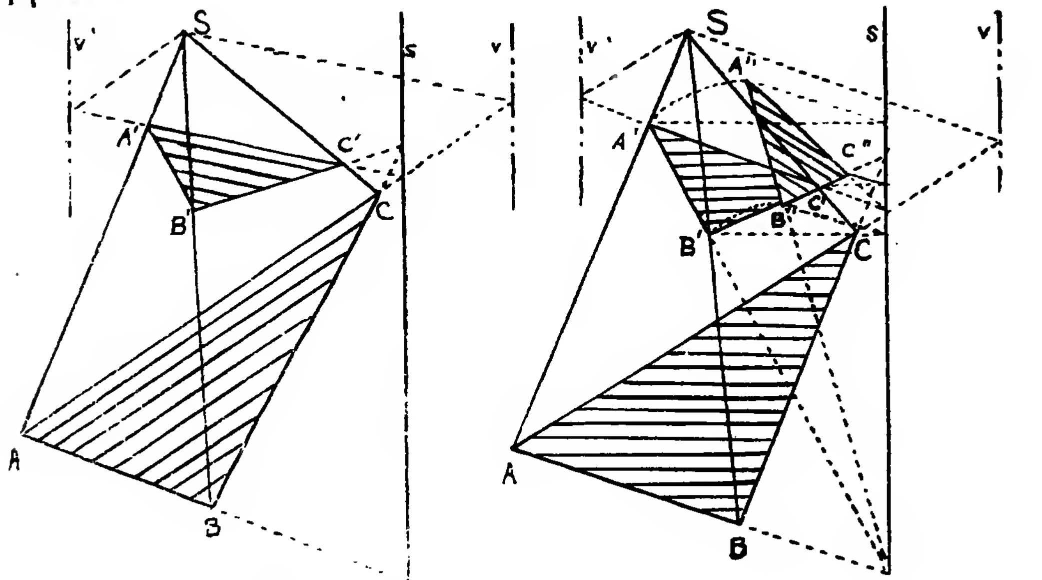
THEOR. 9. *If two figures be in perspective the locus of points in the one figure that have no images in the other is a line parallel to the axis.*

(a) *the two figures not co-planar:*

For the locus of rays that do not pierce a transversal is a plane through the centre parallel to that transversal, and this plane cuts the other transversal in a line parallel to the axis. [Geom.

Q.E.D.

The lines so found in the two transversals are the *vanishing lines* of the two figures



(b) *the two figures co-planar:*

For, turn one figure about the axis to any oblique position, find the vanishing lines as in (a) and turn the figure back to its first position;

then ∵ the same points are images to the same points in the oblique as in the co-planar position,

∴ the same points have no images in the oblique as in the co-planar position.

i.e. the vanishing lines in the oblique position are the vanishing lines in the co-planar position, and they are lines parallel to the axis.

Q.E.D.

COR. I. *If two figures be in perspective, a line of one figure meets the ray parallel to its image, on the vanishing line of the figure.*

For let a be any line of one figure, a' its image, r a ray parallel to a';
then ∵ the image of the point ra is the point ra', [df. proj.
and ∵ r, a' are parallel, [hyp
∴ the point ra has no image. Q.E.D.

COR. 2. *If two figures be in perspective, and if two lines of one figure meet upon the vanishing line of that figure, their images are parallel; and, conversely, if two lines of one figure be parallel, their images in the other figure meet upon the vanishing line.*

It is convenient to speak of the no-point of meeting of two parallel lines as a *point-at-infinity*. The locus of points at infinity in one figure is therefore the image of the vanishing line in the other figure; and that locus is the *line-at-infinity*, of the figure.

COR. 3. *If one vanishing line be at infinity, so is the other also.*

COR. 4. *If the centre of perspective be at infinity, then all rays are parallel.*

This projection is *parallel projection*.

COR. 5. *If the axis of perspective be at infinity, then every line is parallel to its image.*

Th. – 21.

This projection is *homothetic projection*, and the two figures are similar and similarly placed.

COR. 6. *If the projection be both parallel and homothetic* ...

two figures are equal.

THEOR 10 *If two figures in perspective have a vanishing line at infinity, the projection is either parallel or homothetic; and, conversely.*

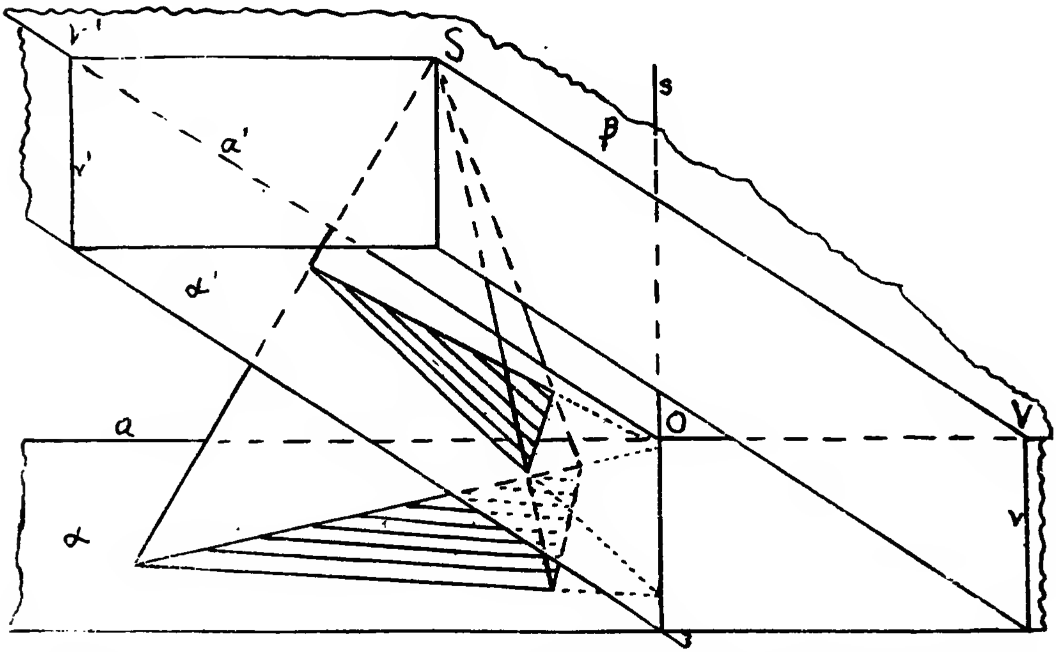
For, if the centre be at infinity, the projection is parallel. [df. par. proj.

But if the centre be not at infinity;
then ∵ a plane through the centre parallel to one transversal meets the other in a line at infinity, [hyp., df. van. line.
∴ the two transversals are parallel, and the axis is at infinity.
∴ the projection is homothetic. Q.E.D.

conversely, if the centre or the axis be at infinity, a plane through the centre parallel to either transversal is at infinity, or it is parallel to both transversals. Q.E.D.

THEOR. 11. *If one of two figures in perspective revolve about the axis, the locus of the centre of perspective is a circle perpendicular to the axis, whose centre is in the vanishing line of the fixed figure.*

For, let α α' be two plane figures in perspective, S the centre, s the axis, and through S pass a plane β perpendicular to s at O, and meeting α, α' in the lines a, a',
then if α' revolve about s, a' remains in the plane β, [Geom.
and ∵ a, a' remain images of each other, [th. 6, cr. 1.
∴ S remains in the plane a, a' i.e. in β.



Through any position of S draw lines SV', SV, parallel to a, a', forming the parallelogram SVOV';
then ∵ V is the co-point of a and the vanishing line of α,
and V' is the co-point of a' and the vanishing line of α', [th. 11 cr. 1.
∴ V, V' are fixed points of α, α', and as α' revolves, S remains at the constant distance VS, = OV, from V. Q.E.D.

EXAMPLES.

1. In ex. 1, 2, 3 of § 3 construct the vanishing lines.

2. In ex. 1, 2, 3 of § 3 let the last $\left\{\begin{matrix}\text{point}\\\text{line}\end{matrix}\right.$ in the data be $\left\{\begin{matrix}V\\v\end{matrix}\right.$ a vanishing $\left\{\begin{matrix}\text{point}\\\text{line}\end{matrix}\right.$ of $\left\{\begin{matrix}ABC\ldots\\abc\ldots\end{matrix}\right.$, and construct the image $\left\{\begin{matrix}A'B'C'\ldots\\a'b'c'\ldots\end{matrix}\right.$ and the vanishing lines.

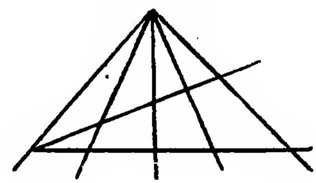
3. in ex. 2 let the given figure be a circle.

4. Through a given point to draw a line parallel to a given line by aid of a given parallelogram.

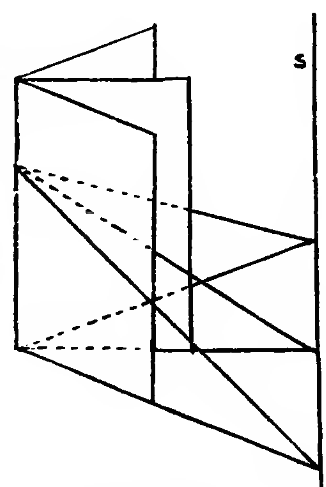
Let a be parallel to b and c to d and let V, s be the given point and line; from s reflect the triangle abc into any convenient triangle a'b'c' with the vertex a'b' at V, and construct the vanishing line V, c'd'.

RANGES AND PENCILS IN PERSPECTIVE.

Some of the most important of the foregoing principles, so far as they apply to the special case of ranges and pencils in perspective, are grouped together below:



If a pencil be cut by two transversals, the sections are ranges in perspective; and, conversely, if two ranges be in perspective, the axis of perspective may be any line through the co-point of the axes of the ranges.



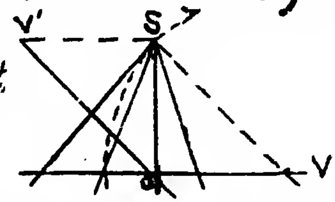
If a book be cut by two transversals the sections are pencils in perspective; and conversely, if two pencils be in perspective the centre of perspective may be any point on the edge of the projecting book.

The images on any transversal of two ranges in perspective from a centre in their plane are co-axial ranges in homology.

The images on any transversal of two concentric non-coplanar pencils in perspective are concentric pencils in homology.

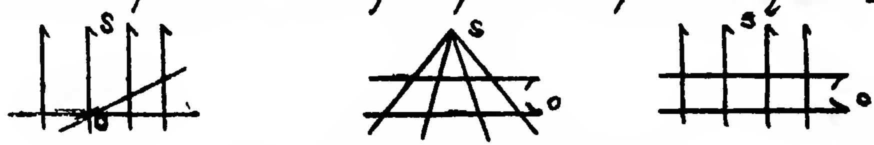
In two {co-axial ranges / concentric pencils} in homology, there are two double {points / rays} and no more, viz the images of the {centre / axis} of perspective and the {co-point / edge} of the {two ranges. / projecting book} [th. 5.

If two ranges be in perspective, the rays parallel to the axes of the ranges meet them in the vanishing points; and if one range turn about the double point, the centre of perspective traces a circle about the fixed vanishing point. [th. 11.



If two ranges be in perspective the mid-point of the two vanishing points is the mid-point of the double points and the centre of perspective.

If either the double point of two ranges in perspective, or the centre of perspective, be at infinity, the ranges are similar; and if both these points be at infinity, the ranges are equal. [Geom.



§5. PROJECTIVE FIGURES.

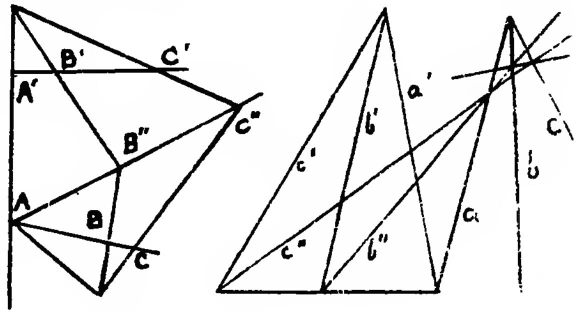
If one figure be projected into another by a sheaf, this second figure into a third by another sheaf, and so on, the series of figures so formed are *projective figures*, and each of them is an image of every other.

A special case is that of a pencil and the range cut from it by any transversal: from its own centre and by its own rays, the pencil is projected into the range; and, conversely, the range is projected into the pencil.

In showing whether two figures be projective, the two figures may be projected into the same plane, and the work thereafter concerns co-planar figures only.

THEOR. 12. *Any two* { *three-point ranges* / *three-ray pencils* } *are projective figures.*

For, let $\left\{\begin{matrix} ABC, A'B'C' \\ abc, a'b'c' \end{matrix}\right.$ be any two $\left\{\begin{matrix} \text{ranges} \\ \text{pencils} \end{matrix}\right.$ and from any $\left\{\begin{matrix} \text{centre upon } AA' \text{ project } A'B'C' \\ \text{axis through } aa' \text{ reflect } a'b'c' \end{matrix}\right.$ into a $\left\{\begin{matrix} \text{range } AB''C''; \\ \text{pencil } ab''c''; \end{matrix}\right.$ then $\because$ the $\left\{\begin{matrix} \text{ranges } AB''C'', ABC \\ \text{pencils } ab''c'', abc \end{matrix}\right.$ are in



perspective, with $\left\{\begin{matrix} BB''-CC'' \text{ as centre,} \\ bb''-cc'' \text{ as axis,} \end{matrix}\right.$

$\therefore \left\{\begin{matrix} ABC, A'B'C' \\ abc, a'b'c' \end{matrix}\right.$ are projective figures. Q.E.D.

COR. 1. *Two figures each consisting of a* $\left\{\begin{matrix} \textit{three-point range} \\ \textit{three-ray pencil} \end{matrix}\right.$ *and a* $\left\{\begin{matrix} \textit{point} \\ \textit{line} \end{matrix}\right.$ *without, are projective figures.*

THEOR. 13. *Any two* $\left\{\begin{matrix} \textit{four-points} \\ \textit{four-lines} \end{matrix}\right.$ *no three of whose* $\left\{\begin{matrix} \textit{points} \\ \textit{lines} \end{matrix}\right.$ *are* $\left\{\begin{matrix} \textit{colinear} \\ \textit{concurrent} \end{matrix}\right.$ *are projective figures.*

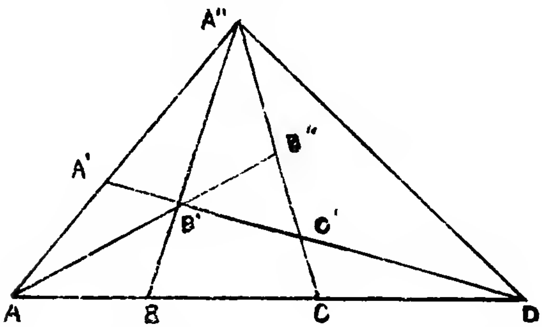
For, let $\left\{\begin{matrix} ABCD, A'B'C'D' \\ abcd\ a'b'c'd' \end{matrix}\right.$ be any two such $\left\{\begin{matrix} \text{four-points,} \\ \text{four-lines,} \end{matrix}\right.$ and let $\left\{\begin{matrix} EFG \\ efg \end{matrix}\right.$ be the diagonal $\left\{\begin{matrix} \text{points} \\ \text{lines} \end{matrix}\right.$ and $\left\{\begin{matrix} H, K \\ h, k \end{matrix}\right.$ the $\left\{\begin{matrix} \text{points } BD-EF, AC-EF \\ \text{lines } bd-ef, ac-ef \end{matrix}\right.$ and $\left\{\begin{matrix} E', F', G', H', K' \\ e', f', g', h', k' \end{matrix}\right.$ the like $\left\{\begin{matrix} \text{points} \\ \text{lines} \end{matrix}\right.$ of $\left\{\begin{matrix} A'B'C'D'. \\ a'b'c'd'. \end{matrix}\right.$ Project $\left\{\begin{matrix} A'B'C'D' \\ a'b'c'd' \end{matrix}\right.$ into $\left\{\begin{matrix} A''B''C''D'' \\ a''b''c''d'' \end{matrix}\right.$ so that the $\left\{\begin{matrix} \text{three-point range } E'F'H' \\ \text{three-ray pencil } e'f'h' \end{matrix}\right.$ projects into the $\left\{\begin{matrix} \text{three-point range } EFH; \\ \text{three-ray pencil } efh; \end{matrix}\right.$

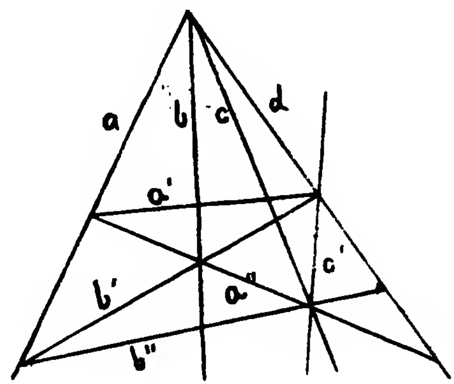
then $\because \left\{\begin{array}{l} K' \text{ projects} \\ k' \text{ reflects} \end{array}\right.$ into $\left\{\begin{array}{l} K \\ k \end{array}\right.$ [th.8.

and $\left\{\begin{array}{l} ABCD,\ A''B''C''D'' \\ abcd,\ a''b''c''d'' \end{array}\right.$ are in perspective, $\left[\begin{array}{l} \text{th.6.} \\ \text{th.7.} \end{array}\right.$

$\therefore \left\{\begin{array}{l} ABCD,\ A'B'C'D' \\ abcd,\ a'b'c'd' \end{array}\right.$ are projective. Q.E.D.

THEOR. 14. *Any $\left\{\begin{array}{l} \text{four-point range } ABCD \\ \text{four-ray pencil } abcd \end{array}\right.$ is projective with the $\left\{\begin{array}{l} \text{ranges} \\ \text{pencils} \end{array}\right.$ found by interchanging the letters in pairs viz* $\left\{\begin{array}{l} BADC,\ CDAB,\ DCBA. \\ badc,\ cdab,\ dcba. \end{array}\right.$





For $\left\{\begin{array}{l} ABCD \text{ projects} \\ abcd \text{ reflects} \end{array}\right.$ from any $\left\{\begin{array}{l} \text{centre } A'' \\ \text{axis } a'' \end{array}\right.$ into $\left\{\begin{array}{l} A'B'C'D, \\ a'b'c'd, \end{array}\right.$ and this from the $\left\{\begin{array}{l} \text{centre } A \\ \text{axis } a \end{array}\right.$ into $\left\{\begin{array}{l} A''B''C'C, \\ a''b''c'c, \end{array}\right.$ and this from the $\left\{\begin{array}{l} \text{centre } B' \\ \text{axis } b' \end{array}\right.$ into $\left\{\begin{array}{l} BADC; \\ badc; \end{array}\right.$ and so for the rest. Q.E.D.

THEOR. 15. *Two equal figures are projective figures.*

For one figure may be made to coincide with the other by revolving it about three axes:

1. The co-line of their planes, taking care that the figure be so turned that it comes into the plane of the fixed figure with like parts of the two figures in the same order.

2. The external mid-line of two like lines of the two figures, bringing these two lines into coincidence, with like segments lying in opposite directions.

3. A perpendicular to the coincident like lines through the mid-point of two like points.

And the figure so revolved, in all its positions, is in perspective with that which precedes and that which follows it. Q.E.D. [Th.6.

COR. *Two similar figures are projective.*

THEOR. 16. *If a moving* $\left\{\begin{array}{l}\textit{point trace}\\ \textit{line envelop}\end{array}\right.$ *a plane curve, an image of this* $\left\{\begin{array}{l}\textit{point}\\ \textit{line}\end{array}\right.$ *in any plane* $\left\{\begin{array}{l}\textit{traces}\\ \textit{envelops}\end{array}\right.$ *a curve that is an image of the first; and the* $\left\{\begin{array}{l}\textit{tangent at}\\ \textit{contact of}\end{array}\right.$ *the moving* $\left\{\begin{array}{l}\textit{point}\\ \textit{line}\end{array}\right.$ *and that* $\left\{\begin{array}{l}\textit{at}\\ \textit{of}\end{array}\right.$ *the image are images of each other.*

For, let $\left\{\begin{array}{l}P\\ p\end{array}\right.$ be the moving $\left\{\begin{array}{l}\text{point}\\ \text{line}\end{array}\right.$ and $\left\{\begin{array}{l}P'\\ p'\end{array}\right.$ an image from any centre S upon any plane, then as $\left\{\begin{array}{l}P\text{ traces}\\ p\text{ envelops}\end{array}\right.$ a curve the $\left\{\begin{array}{l}\text{line }SP\text{ traces}\\ \text{plane }Sp\text{ envelops}\end{array}\right.$ a cone, and $\left\{\begin{array}{l}P'\text{ traces}\\ p'\text{ envelops}\end{array}\right.$ the section of this cone by the plane in which $\left\{\begin{array}{l}P'\\ p'\end{array}\right.$ moves;

$\therefore$ it is an image of the section $\left\{\begin{array}{l}\text{traced by }P.\\ \text{enveloped by }p.\end{array}\right.$

And the $\left\{\begin{array}{l}\text{tangent plane thro }SP\text{ to}\\ \text{element of contact }Sp\text{ with}\end{array}\right.$ the cone $\left\{\begin{array}{l}\text{cuts}\\ \text{pierces}\end{array}\right.$ the two transversals in images that are the $\left\{\begin{array}{l}\text{tangents at }P, P'.\\ \text{contacts of }p, p'.\end{array}\right.$

So for projections from successive centres. Q.E.D.

The greatest number of $\left\{\begin{matrix}\text{points on}\\ \text{tangents to}\end{matrix}\right.$ a curve that $\left\{\begin{matrix}\text{lie upon}\\ \text{pass thro}\end{matrix}\right.$ a single $\left\{\begin{matrix}\text{line}\\ \text{point}\end{matrix}\right.$ shows the $\left\{\begin{matrix}\text{order}\\ \text{class}\end{matrix}\right.$ of the curve.

e.g. a circle is a curve of the second order and the second class.

COR. *A curve and its images are of the same order and of the same class.*

The images of a circle are *conics*.

If the vanishing line lie without the circle, the image has no point at infinity, and is an *ellipse*.

If the vanishing line touch the circle, the image has one real point at infinity, the image of the point of contact, and is a *parabola*. All lines through the point of contact project into parallel lines whose direction is the *direction to infinity* of the parabola. The tangent line, being tangent to the circle projects into a tangent to the parabola at infinity, and is the *line at infinity* of the parabola.

If the vanishing line cut the circle, the image has two real points at infinity, the images of the co-points of the vanishing line and the circle, and is an *hyperbola*. The lines through these two points at infinity form two systems of parallel lines whose directions are the directions to infinity of the hyperbola. The two tangents to the circle at these co-points, project into tangents to the hyperbola at infinity, and these are the ***asymptotes*** of the hyperbola.

A figure formed of two lines or two points is an *improper conic*: in a limited sense these figures are the projective images of a circle and they possess many of the properties common to all conics.

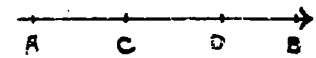
Points within a circle project into points *within the conic*, and points without a circle project into points *without the conic*.

§6. CROSS RATIOS.

Every line is assumed to be directed, and segments of a line that reach in the direction of the line are positive segments; but segments that reach in the opposite direction are negative segments.

E.g., in the first figure AB, AC, AD, BC, BD, CD are all positive.

in the second figure AB, AC, AD, CD are positive; BC, BD are negative.



in the third figure AB, AC, AD, BC are negative, BD, CD are positive.

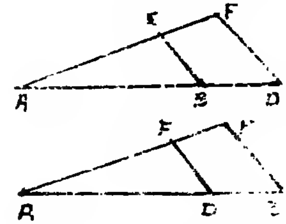
If a segment of a line be cut at any point, the segments so formed reach from the initial point of the given segment to the dividing point, and from this point to the terminal point; and the *ratio of division* is the ratio of the first segment to the other.

E.g., if AD be cut at B, the segments are AB, BD, and the ratio of division is AB:BD.

If the dividing point lie on the given segment, the two segments have the same sign and their ratio of division is positive; if they lie on the given segment produced, they are of opposite signs, and the ratio is negative, if it lie at the initial point of the segment, the ratio is zero, if at the mid-point, the ratio is unity; if at the terminal point, the ratio is infinity; if at infinity in either direction, the ratio is unity and negative.

THEOR. 17. *A segment can be cut in a given ratio in one point and but one.*

For, let AD be a given segment; through A draw a line, and upon



it take AE, EF, segments in the given ratio; join FD and draw EB parallel to FD cutting AD in B;

then B satisfies the condition, and is the point sought,

and there is no other such point,

for no line through E not parallel to FD can cut AD in the given ratio.

Q.E.D.

If a segment of a line be divided at two points, thus forming a four-point range whose extremes are the ends of the segment, and whose means are the points of division, then the ratio of the two ratios of division is the *cross-ratio* of the range.

E.g. if AD be divided at B, C, then AB : BD, AC : CD are ratios of division, and $\overline{AB:BD} : \overline{AC:CD}$, or its equal $\overline{AB\cdot CD} : \overline{AC\cdot BD}$, is the cross-ratio of the four-point range ABCD. It is written (ABCD).

Since the four letters A, B, C, D may be permuted in twenty four ways, there are twenty four cross-ratios for the same four points:

(ABCD) +	(CBAD) = 1	(BCAD) +	(ACBD) = 1	(CABD) +	(BACD) = 1
(BADC)	(DABC)	(CBDA)	(BDAC)	(DBAC)	(CDBA)
(CDAB)	(ADCB)	(DACB)	(CADB)	(ACDB)	(DCAB)
(DCBA)	(BCDA)	(ADBC)	(DBCA)	(BDCA)	(ABDC)

THEOR. 18. *A segment of a line is cut by the ends of another segment of the same line in the same cross-ratio as the second segment is cut by the ends of the first, and in the same cross-ratio as the segments reversed cut each other.*

Like Th...

For, $(ABCD) \equiv \overline{AB\cdot CD} : \overline{AC\cdot BD} = \overline{BA\cdot DC} : \overline{BD\cdot AC} \equiv (BADC)$.

$= \overline{CD\cdot AB} : \overline{CA\cdot DB} \equiv (CDAB)$.

$= \overline{DC\cdot BA} : \overline{DB\cdot CA} \equiv (DCBA)$ Q.E.D.

NOTE. In the above table of cross-ratios of the four points A, B, C, D, each of

the six columns consists of a set of four equal cross-ratios; and, in general, ratios from different columns are unequal.

THEOR. 19. *If A, B, C, D be four colinear points, then*
$AB\cdot CD + BC\cdot AD + CA\cdot BD = 0$, *and the six ratios between the three terms of this symmetric relation, taken two and two, are the opposites of the six unequal cross-ratios of the four points.*

$$\begin{aligned}
\text{For} \because\ AB\cdot CD &= (AD+DB)\cdot CD = AD\cdot CD - BD\cdot CD\\
BC\cdot AD &= (BD+DC)\cdot AD = BD\cdot AD - CD\cdot AD\\
CA\cdot BD &= (CD+DA)\cdot BD = CD\cdot BD - AD\cdot BD\\
\therefore\ AB\cdot CD &+ BC\cdot AD + CA\cdot BD = 0.
\end{aligned}$$

Anharmonic Ratio Q.E.D.

and
$$\begin{aligned}
&\overline{AB\cdot CD} : \overline{CA\cdot BD} = -(ABCD), \quad \overline{CA\cdot BD} : \overline{AB\cdot CD} = -(ACBD),\\
&\overline{BC\cdot AD} : \overline{CA\cdot BD} = -(CBAD), \quad \overline{CA\cdot BD} : \overline{BC\cdot AD} = -(CABD),\\
&\overline{BC\cdot AD} : \overline{AB\cdot CD} = -(BCAD), \quad \overline{AB\cdot CD} : \overline{BC\cdot AD} = -(BACD).
\end{aligned}$$

Q.E.D.

COR. 1. *If r be any cross-ratio of four colinear points A, B, C, D, the six unequal cross-ratios are:* r, $1-r$, $\overline{r-1}:r$, $1:r$, $1:\overline{1-r}$, $r:\overline{r-1}$.

COR. 2. *If two sets of four points have a cross-ratio of the one set equal to a cross-ratio of the other, the six cross-ratios of the one set are severally equal to the six cross-ratios of the other.*

Harmonic Ratio

COR. 3. *If r be* 1, *the six cross-ratios are* $1, 0, 0, 1, \infty, \infty$; *if r be* -1, *they are* $-1, 2, 2, -1, \frac{1}{2}, \frac{1}{2}$; *and no two of the six cross-ratios of any four points are equal unless they be included in one of these two sets.*

THEOR. 20. *If three points of a range be given, there is one and but one point that completes the range in a given cross-ratio.*

For ∵ the ratio of division of the given segment by the given point is known,

and the cross-ratio is known,

∴ the ratio of division of the given segment by the point sought is known,

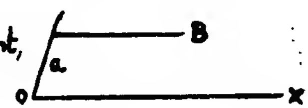
and there is but one such point. Q.E.D. Th. 17

COR. *If the cross-ratios of four colinear points be 1, 0, ∞, then two of the four points coincide.*

The *distance* from a point to a line is the length of a segment of a line parallel to a fixed line and reaching from the point to the line. The fixed line is the *axis of measurement* for the given line, and the distance is positive / negative when measured with / against the direction of the axis.

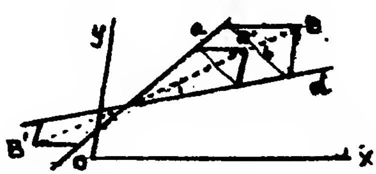
So, the distance from a line to a point is the length of the reversed segment, reaching from the line to the point.

E.g., in the figure, if oX be the axis of measurement, the distance aB is positive and Ba is negative.



If the axis of measurement be perpendicular to the given line, the distance is *the perpendicular distance.*

If ox, oy be the axes of the lines a, d, then any point B divides the angle in the ratio aB : Bd.



<u>THEOR. 21.</u> *With given axes of measurement, all points on a line through the co-point of two lines divide their angle in the same ratio; and, conversely.*

Let a, d be any two lines, b a line through their co-point, B, B′ points on b,

ox, oy any axes of measurement of a, d,

then either aB, aB′ are of the same sign, and so are Bd, B′d,

or else aB, aB′ are of opposite signs, and so are Bd, B′d,

and in either case $aB : Bd = aB' : B'd$. Q.E.D. [sim. tri.

Conversely, if B, B' divide $\angle\, ad$ so that $aB : Bd = aB' : B'd$

then B, B' lie within the same angle,

for not other-wise may $\overline{aB : Bd}, \overline{aB' : B'd}$ have the same sign;

and they lie on a line through point ad.

For $\because$ aB is parallel to aB' and Bd to $B'd$.

and $aB : Bd = aB' : B'd$ [Hyp.

$\therefore$ the triangles so formed are similar and in homothetic perspective,

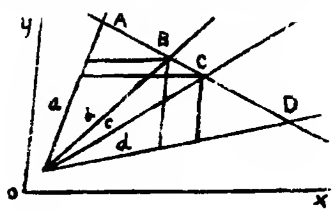
$\therefore$ the points, B, B' are colinear with the co-point of a, d. Q.E.D

If a line divide an angle, the *ratio of division* by that line, as to given axes of measurement, is the ratio of division of the angle by any point of the line.

The *cross-ratio of a pencil* of four rays is the ratio of the two ratios in which the angle of the two extreme rays is divided by the means.

exam.

THEOR. 22. *The cross-ratio of a four-ray pencil is equal to the cross-ratio of any four-point range cut from it, and is independent of the axes of measurement.*



For, let $a\,b\,c\,d$ be a four-ray pencil cut by a transversal in the range $ABCD$;

then $\because$ $aB : aC = AB : AC$, $Cd : Bd = CD : BD$

$\therefore$ $\overline{aB \cdot Cd} : \overline{aC \cdot Bd} = \overline{AB \cdot CD} : \overline{AC \cdot BD}$

i.e. the cross-ratio of the pencil $= (ABCD)$;

and $(ABCD)$ is the same whatever the axes of measurement. Q.E.D.

COR. 1. *The cross-ratio of a four-ray pencil is not changed if the rays be interchanged in pairs.*

COR. 2. *If three rays of a pencil be given, there is one ray and but one that completes the pencil in a given cross-ratio.*

Cor. 3. *The cross-ratios of any two projective figures of four elements (ranges or pencils) are equal.*

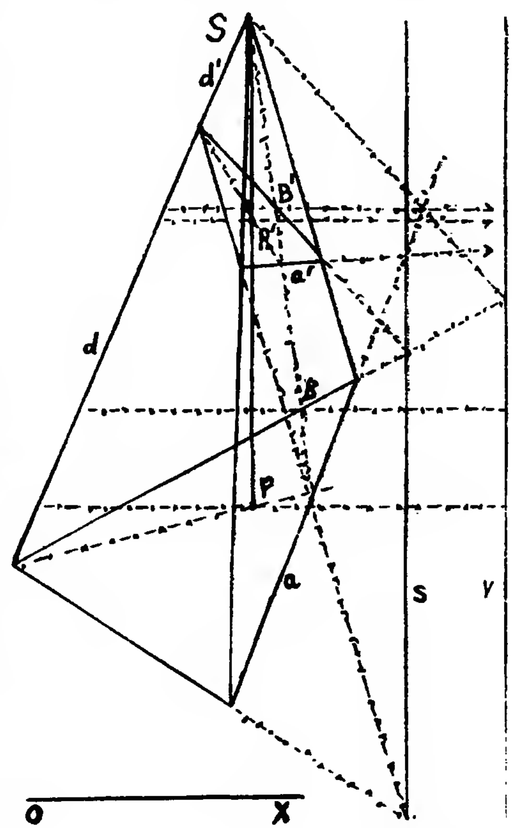
Theor. 23. *If there be two projective figures and if an angle in one of them be divided by a variable point, the ratio of division bears a constant ratio to the like ratio of division in the other figure.*

For, let a, d be two lines, B a fixed point and P a variable point and let a', d', B', P' be the images of a, d, B, P;

then $\because (aBPd) = (a'B'P'd')$ [th. 22, cr. 3

$\therefore \overline{aP:Pd} : \overline{a'P':P'd'} = \overline{aB:Bd} : \overline{a'B':B'd'}$, a constant. Q.E.D.

Cor. *If two figures be in homology the distance of a variable point in one of them from its vanishing line varies as the ratio of the distances of this point and its image from the centre of homology.*



Let S be the centre of homology, d, d' a pair of coincident rays, a, a' any pair of like lines; B, B' a pair of fixed points, P, P' a pair of variable points, v the vanishing line of figure a, d;

then $\because Pd : P'd' = SP : SP'$ and $Bd : B'd' = SB : SB'$ [sim. tri.

and $\overline{aP:aB} : \overline{a'P':a'B'} = \overline{Pd:P'd'} : \overline{Bd:B'd'}$, [th.

$\therefore \overline{aP:aB} : \overline{a'P':a'B'} = \overline{SP:SP'} : \overline{SB:SB'}$.

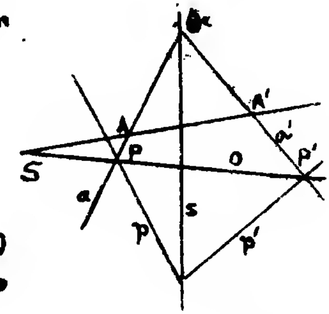
Let a be v;

then $\because a'$ is at infinity and $a'P' : a'B' = 1$

$\therefore vP : vB = \overline{SP:SP'} : \overline{SB:SB'}$. Q.E.D.

OMIT

THEOR. 24. *If two figures be in homology the axis and centre of homology divide the* $\left\{\begin{array}{l}\textit{segment}\\ \textit{angle}\end{array}\right.$ *between any pair of like* $\left\{\begin{array}{l}\textit{points}\\ \textit{lines}\end{array}\right.$ *in a constant cross-ratio.*



For, let S, s, be the centre and axis of two figures in homology; $\left\{\begin{array}{l}A, A', P, P'\\ a, a', p, p'\end{array}\right.$ two pairs of like $\left\{\begin{array}{l}\text{points,}\\ \text{lines,}\end{array}\right.$

then $\because$ $\left\{\begin{array}{l}AP, A'P' \text{ reflect}\\ ap, a'p' \text{ project}\end{array}\right.$ into each other from $\left\{\begin{array}{l}s \text{ at } O\\ S \text{ at } o.\end{array}\right.$

$\therefore$ the $\left\{\begin{array}{l}\text{ranges } ASsA', PSsP'\\ \text{pencils } aSsa', pSsp'\end{array}\right.$ are in perspective from $\left\{\begin{array}{l}O\\ o\end{array}\right.$

and $\left\{\begin{array}{l}(PSsP') = (ASsA'),\\ (pSsp') = (aSsa'),\end{array}\right.$ a constant. Q.E.D.

omit.

COR. 1. *The cross-ratio in which the axis and centre divide the segment of a pair of like points is equal to the cross-ratio in which they divide the angle of a pair of like lines.*

For, let A, A' be a pair of like points on the like lines a, a';

then $\because$ the range $ASsA'$ is a section of the pencil $aSsa'$

$\therefore$ $(ASsA') = (aSsa')$ Q.E.D.

This constant cross-ratio is the *parameter* of the homology.

EXAMPLES.

Do-out. Exam.

1. Given three $\left\{\begin{array}{l}\text{points}\\ \text{rays}\end{array}\right.$ of a $\left\{\begin{array}{l}\text{range,}\\ \text{pencil,}\end{array}\right.$ to complete the $\left\{\begin{array}{l}\text{range}\\ \text{pencil}\end{array}\right.$ in a given cross-ratio.

2. If there be two triangles, and if a fixed $\left\{\begin{array}{l}\text{point}\\ \text{line}\end{array}\right.$ and a variable $\left\{\begin{array}{l}\text{point}\\ \text{line}\end{array}\right.$ in the plane each of them be so related that these $\left\{\begin{array}{l}\text{points}\\ \text{lines}\end{array}\right.$ divide two $\left\{\begin{array}{l}\text{angles}\\ \text{sides}\end{array}\right.$ of their triangle pairs of equal cross-ratios, then the two figures described by the two variable

{points / lines} are projective; and conversely.

3. If there be two sets of four coplanar {points / lines} paired in any given way, then for a given point or line in the one figure the like point or line in the other figure is determined.

4. Two projective figures such that a {four-point / four-line} of the one is in perspective with its image in the other are in perspective throughout.

5. Two coplanar projective figures, such that three {points / lines} of the one figure are their own images in the other, are in homology; and if there be four {points, / lines,} no three of which are {colinear, / concurrent,} that are their own images, the two figures coincide throughout.

6. To construct a figure in homology with a given figure, given the centre, axis, and parameter.

§7. HARMONIC RATIOS.

If a {segment / angle} be divided internally and externally in opposite ratios of division, it is divided harmonically, and the cross-ratio, -1, so formed is a *harmonic ratio*. The two {points / lines} of division are *conjugate* {*points.* / *lines.*}

E.g. The conjugate of {the mid-point of a segment is the point at infinity on the segment produced. / a mid-line of an angle is the other mid-line.}

THEOR. 25. *If a {segment / angle} be divided harmonically by the {ends / sides} of*

another $\left\{\begin{matrix}\textit{segment}\\ \textit{angle}\end{matrix}\right.$ $\left\{\begin{matrix}\textit{of}\\ \textit{at}\end{matrix}\right.$ *the same* $\left\{\begin{matrix}\textit{line,}\\ \textit{point}\end{matrix}\right.$ *the second* $\left\{\begin{matrix}\textit{segment}\\ \textit{angle}\end{matrix}\right.$ *is divided harmonically by the* $\left\{\begin{matrix}\textit{ends}\\ \textit{sides}\end{matrix}\right.$ *of the first* $\left\{\begin{matrix}\textit{segment,}\\ \textit{angle,}\end{matrix}\right.$ *and either* $\left\{\begin{matrix}\textit{segment}\\ \textit{angle}\end{matrix}\right.$ *is divided harmonically by the other reversed.* [th. df.

COR. *The six cross-ratios of the four points of a harmonic range are:* $-1, 2, 2, -1, \frac{1}{2}, \frac{1}{2}$.

THEOR. 26. *In a harmonic range the distance from any point to its conjugate is the harmonic mean of the distances from this point to the other two points.*

For, let A be the point, D its conjugate, B, C, points dividing the segment AD harmonically, Known by the Greek

then $\because$ $AB\cdot CD = -AC\cdot BD$ [df. har. div.

$\therefore$ $AB(AD-AC) = -AC(AD-AB)$

$\therefore$ $AD = 2AB\cdot AC : \overline{AB+AC}$. Q.E.D.

THEOR. 27. *In a harmonic range the distance from the mid-point of a pair of conjugate points to either of them is the geometric mean of the distances from this mid-point to the other two points.*

For, let O be the mid-point of the segment AD, and B, C points dividing AD harmonically,

then $\because$ $AB\cdot CD = -AC\cdot BD$.

$\therefore$ $(AO+OB)(OD-OC) = (AO+OC)(OB-OD)$,

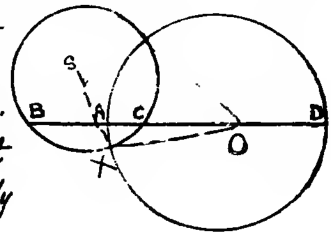
i.e. $(OD+OB)(OD-OC) = (OD+OC)(OB-OD)$,

$\therefore$ $OD^2 = OB\cdot OC$. Q.E.D.

COR. 1. *The two segments of a harmonic range overlap each other, but neither reaches beyond the mid-point of the other.*

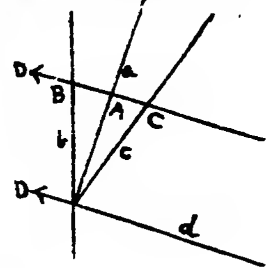
For $\because$ the product $OB \cdot OC = OD^2$, and is positive,

$\therefore$ either B or C lies between O, D and both lie on the same side of O. Q.E.D.



COR 2. *A circle through the ends of either segment of a harmonic range cuts at right angles the circle upon the other segment as diameter; and, conversely, if two circles cut each other at right angles, either of them divides harmonically the diameter of the other.*

THEOR. 28. *If in a harmonic pencil a pair of conjugate rays be at right angles, they are the mid-lines of the other pair.*



For, cut the harmonic pencil $abcd$ by a transversal perpendicular to a, giving the harmonic range $ABCD$;

then $\because$ d is perpendicular to a, and so is the transversal,

$\therefore$ D is at infinity, and A is a mid-point of BC,

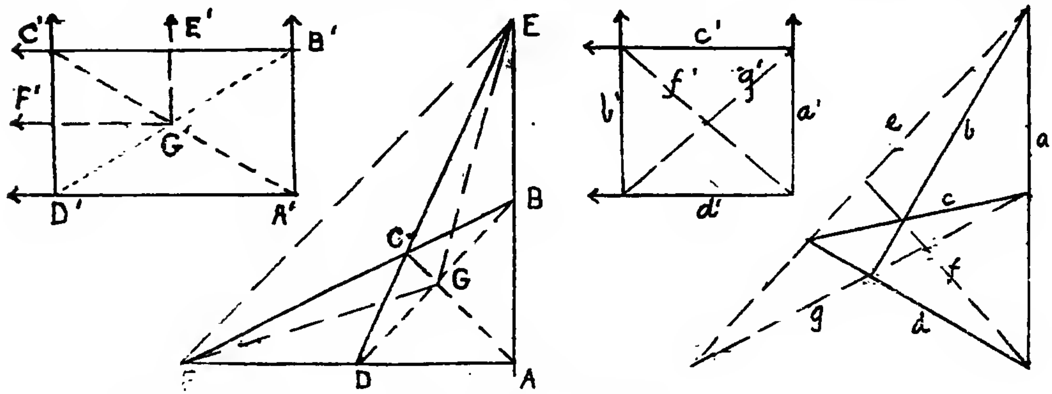
$\therefore$ a is one mid-line of $b\,c$, and d is the other.

Q.E.D.

THEOR. 29. *Two opposite* $\left\{\begin{matrix}\textit{sides}\\ \textit{vertices}\end{matrix}\right.$ *of a complete* $\left\{\begin{matrix}\textit{four-point}\\ \textit{four-line}\end{matrix}\right.$ *and the two* $\left\{\begin{matrix}\textit{sides}\\ \textit{vertices}\end{matrix}\right.$ *of a diagonal triangle that are* $\left\{\begin{matrix}\textit{concurrent}\\ \textit{colinear}\end{matrix}\right.$ *with them form a harmonic* $\left\{\begin{matrix}\textit{pencil.}\\ \textit{range.}\end{matrix}\right.$

For, let $\left\{\begin{matrix}ABCD\\ abcd\end{matrix}\right.$ be any $\left\{\begin{matrix}\text{four-point, } EFG\\ \text{four line, } efg\end{matrix}\right.$ the diagonal triangle;

project this figure into the rectangle $\left\{\begin{matrix}A'B'C'D';\\ a'b'c'd';\end{matrix}\right.$



then $\because \left\{\begin{matrix}\text{lines } G'E', G'F' \\ \text{points } g'e', g'f'\end{matrix}\right.$ bisect the $\left\{\begin{matrix}\text{angles } A'C'\text{-}B'D', \\ \text{segments } a'c'\text{-}b'd',\end{matrix}\right.$

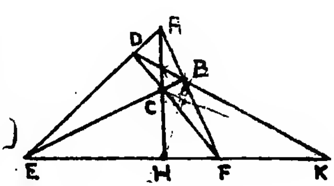
$\therefore$ the $\left\{\begin{matrix}\text{pencil } A'C', G'E', G'F', B'D' \\ \text{range } a'c', g'e', g'f', b'd'\end{matrix}\right.$ is harmonic

$\therefore$ the projective $\left\{\begin{matrix}\text{pencil } AC, GE, GF, BD \\ \text{range } ac, ge, gf, bd\end{matrix}\right.$ is harmonic. Q.E.D.

Cor. *The $\left\{\begin{matrix}\text{sides} \\ \text{angles}\end{matrix}\right.$ of a $\left\{\begin{matrix}\text{four-point} \\ \text{four-line}\end{matrix}\right.$ are divided harmonically by the $\left\{\begin{matrix}\text{sides} \\ \text{vertices}\end{matrix}\right.$ of its diagonal triangle; and, conversely.*

Prob. 2. *Given three $\left\{\begin{matrix}\text{points} \\ \text{rays}\end{matrix}\right.$ to construct the fourth $\left\{\begin{matrix}\text{point} \\ \text{ray}\end{matrix}\right.$ of a harmonic $\left\{\begin{matrix}\text{range.} \\ \text{pencil.}\end{matrix}\right.$*

Let E, F, H be the three points; from H draw any convenient line and join E, F, to two points A, C, upon it; and let B, D be the other co-points; then K, the co-point of BD, EF is the point sought. Q.E.F. [th.29.cr.

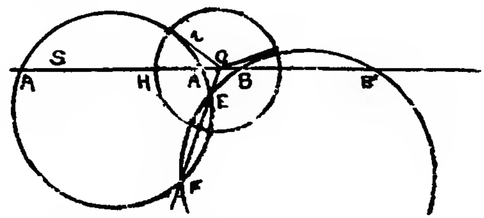


☞ In this problem, and hereafter, when only one of a pair of correlative,

propositions is demonstrated, the other is left as an exercise for the reader.

PROB. 3. *Given two segments of a line, to find two points that shall divide each of them harmonically.*

Let AA′, BB′ be two segments of a line s; upon AA′, BB′ as chords draw two circles c, c′, meeting in E, F; let EF cut s in O, with O as centre and with radius r, equal to the tangent from O to either circle, i.e. such that $r^2 = OE \cdot OF$, draw a circle cutting s in H, K; then H, K are the points sought. Q.E.F. [Th. 27, cr. 2.



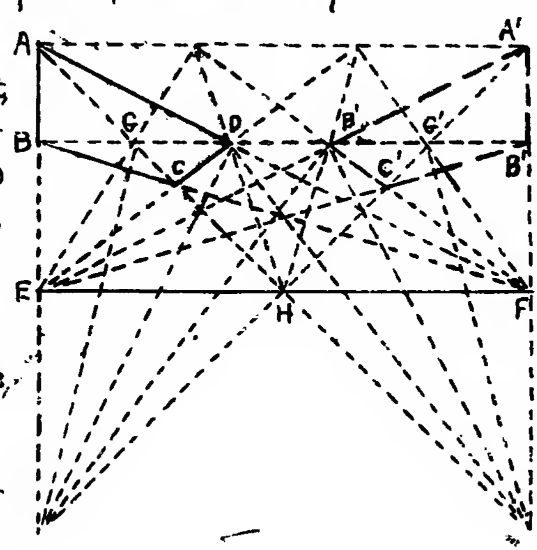
If the segments overlap, the product OE·OF is negative and there is no solution.

EXAMPLES.

1. Given two parallel lines, to bisect a segment of one of them.

2. Given a segment (EF), and its mid-point H, to draw a line parallel to this segment through a given point A.

3. Given a segment and its mid-point, to divide the segment into any number of equal parts; construct a line BD parallel to the given segment and on BD construct any number of equal segments BG, GD, ⋯.



4. Given a circle and its centre, to bisect a given angle.

5. Given an angle and one bisector, to draw the other bisector.

§ 8. PROJECTIVE RANGES AND PENCILS.

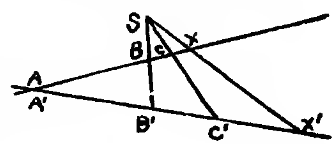
<u>THEOR. 30</u>. *If* $\left\{\begin{matrix} A, A',\ B, B',\ C, C' \cdots \\ a, a',\ b, b',\ c, c' \cdots \end{matrix}\right.$ *be pairs of* $\left\{\begin{matrix}\text{points}\\ \text{rays}\end{matrix}\right.$ *forming two* $\left\{\begin{matrix}\text{ranges}\\ \text{pencils}\end{matrix}\right.$ $\begin{matrix} ABC\cdots,\ A'B'C'\cdots \\ abc\cdots,\ a'b'c'\cdots \end{matrix}$ *wherein the* $\left\{\begin{matrix}\text{ranges } ABCX,\ A'B'C'X' \\ \text{pencils } abcx,\ a'b'c'x'\end{matrix}\right.$ *formed from three pairs of* $\left\{\begin{matrix}\text{points}\\ \text{rays}\end{matrix}\right.$ *and any fourth pair are equicross, then the two* $\left\{\begin{matrix}\text{ranges } ABC\ldots,\ A'B'C'\ldots \\ \text{pencils } abc\ldots,\ a'b'c'\ldots\end{matrix}\right.$ *are projective.*

For, project the first range so that ABC projects into $A'B'C'$ and x into x''

then $\because (ABCX) = (A'B'C'X')$ [hyp.

$= (A'B'C'X'')$ [th. 22, cr. 3.

$\therefore\ X'' \equiv X'$ Q.E.D [th. 20.



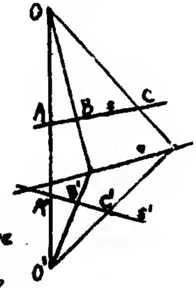
COR. 1. *If in two projective* $\left\{\begin{matrix}\text{ranges}\\ \text{pencils}\end{matrix}\right.$ *three* $\left\{\begin{matrix}\text{points}\\ \text{lines}\end{matrix}\right.$ *of one coincide with their images in the other, the two* $\left\{\begin{matrix}\text{ranges}\\ \text{pencils}\end{matrix}\right.$ *coincide throughout.*

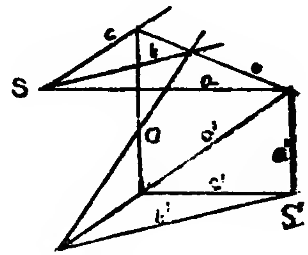
COR. 2. *If in two projective* $\left\{\begin{matrix}\text{ranges}\\ \text{pencils}\end{matrix}\right.$ *a* $\left\{\begin{matrix}\text{point}\\ \text{ray}\end{matrix}\right.$ *of one coincide with its image in the other, the two* $\left\{\begin{matrix}\text{ranges}\\ \text{pencils}\end{matrix}\right.$ *are in perspective.*

For if $A \equiv A'$, then $ABC\cdots$ projects from $S, \equiv BB' - CC'$, into $A'B'C'\cdots$.

COR. 3. *If two projective* $\left\{\begin{matrix}\text{ranges}\\ \text{pencils}\end{matrix}\right.$ *be* $\left\{\begin{matrix}\text{projected from}\\ \text{cut by}\end{matrix}\right.$ *two* $\left\{\begin{matrix}\text{centers}\\ \text{axes}\end{matrix}\right.$ *that are* $\left\{\begin{matrix}\text{colinear}\\ \text{concurrent}\end{matrix}\right.$ *with any* $\left\{\begin{matrix}\text{point}\\ \text{ray}\end{matrix}\right.$ *of one* $\left\{\begin{matrix}\text{range}\\ \text{pencil}\end{matrix}\right.$ *and its image in the other, then the two* $\left\{\begin{matrix}\text{pencils}\\ \text{ranges}\end{matrix}\right.$ *so formed are in perspective; and their* $\left\{\begin{matrix}\text{co-range}\\ \text{co-pencil}\end{matrix}\right.$

is in perspective with each of the given {ranges / pencils}.



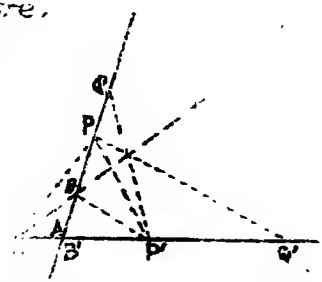


For ∵ OA, O′A′ coincide,

∴ O-ABC⋯, O′-A′B′C′⋯ are in perspective

and their co-range o is in perspective from O with ABC⋯ and from O′ with A′B′C′⋯. Q.E.D.

THEOR. 31. *In two projective {ranges / pencils} the {co-line / co-point} of any pair of {points / rays} taken one from each {range / pencil} and the {co-line / co-point} of their images {reflect / project} into each other from a fixed {axis / centre.}*



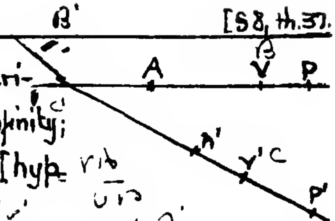
For, let ABPQ⋯, A′B′P′Q′⋯ be projective ranges wherein A coincides with B′, and P, Q′ are any two points one from each range;

then ∵ the pencils P–A′B′P′Q′⋯, P′–ABPQ are in perspective [th. 30 cr. 2.

∴ PQ′, P′Q reflect into each other from the axis of perspective, the co-line of PB′-P′B and PA′-P′A,

i.e. the fixed line A′B.

THEOR. 32. *If a pair of variable points so slide upon fixed axes, that the product of their distances from fixed points on the axes is constant, they generate projective ranges of which the fixed points are the vanishing points; and, conversely, any pair of projective ranges may be so generated.*



For, let V, V' be two fixed points upon the axes, P, P' two variable points, A, A' two fixed positions of P, P', I, I' the two points at infinity;
then ∵ $VA \cdot V'A' = VP \cdot V'P'$, [hyp.

∴ $VA : VP = V'P' : V'A'$,

i.e. $(VAPI) = (I'A'P'V')$

∴ P, P' generates the projective ranges, V, A, I, ⋯, I', A', V', ⋯, Q.E.D. [th. 30.

Conversely, let V, V' be the vanishing points and I, I' the points at infinity of two projective ranges, and let A, A' be a pair of like fixed points and P, P' a pair of variable points;
then ∵ $(VAPI) = (I'A'P'V')$ [th. 22, cr. 3

∴ $VP \cdot V'P' = VA \cdot V'A'$. Q.E.D.

The mid-point of the vanishing points of two coaxial projective ranges is the *centre* of the ranges.

COR. *If in two coaxial projective ranges their centre be taken as a point of one range, it cuts the segment between its image and the vanishing point of the other range: externally, if the ranges have two double points; at its image, if they have but one double point; internally, if they have no double point.*

For, let V, V' be the vanishing points, O their mid-point and O' its image, S a double point;
then ∵ $VO = OV'$, $VO \cdot V'O' = VS \cdot V'S$, [hyp. th.

∴ $VO(V'O + OO') = (VO + OS)(V'O + OS)$,

∴ $VO \cdot OO' = OS^2$

∴ $OV' \cdot OO' = OS^2$

and if V', O' lie on the same side of O, then $OV' \cdot OO'$ is positive, and there are two points S equidistant from O;
if O, O' coincide, then $OV' \cdot OO'$ is zero and S is at O;

if V′, O′ lie on opposite sides of O, then OV′·OO′ is negative, and there is no double point. Q.E.D.

THEOR. 33. *If a pair of variable {points slide / rays turn} in a constant cross-ratio with two fixed {points / rays} they generate a pair of projective {ranges / pencils} whose double {points / rays} are the fixed {points / rays}; and, conversely, any pair of projective {coaxial ranges / concentric pencils} that have two double {points / rays} may be so generated.*

For, let S, T be the fixed points, P, P′ a pair of variable points, and A, A′ any fixed positions of P, P′;

then ∵ (SPP′T) = (SAA′T) [hyp.

∴ (SPAT) = (SP′A′T)

∴ P, P′ generate the projective ranges SAT···, SA′T···,

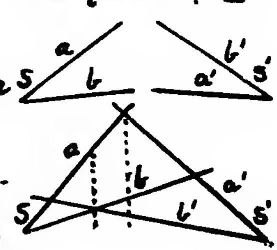
and S, T are double points. Q.E.D.

Conversely, let S, T be the double points of two coaxial ranges in perspective, P, P′ a pair of variable points and A, A′ any fixed positions of P, P′;

then ∵ (SPAT) = (SP′A′T)

∴ (SPP′T) = (SAA′T) Q.E.D.

If of two coplanar pencils, one may be made to coincide with the other by sliding and turning it in its own plane, the two pencils are *equal*; if by turning it over, they are *opposite*.



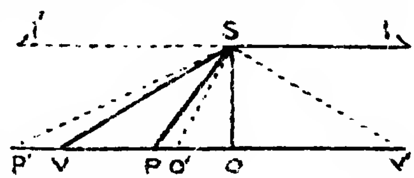
THEOR. 34. *In two equal pencils the angle between a pair of like variable rays is constant; in two opposite pencils the mid-lines of a pair of like rays have a fixed direction.* [Geom.

COR. 1. *Two concentric equal pencils have no double ray, and their sections by a transversal have no double point.*

COR. 2. *Two concentric opposite pencils have two double rays, at right angles to each other, and their sections by a transversal have two double points.*

THEOR. 35. *Every pair of coaxial projective ranges with no double point is a section of a pair of concentric equal pencils.*

For, let V, V' be the vanishing points of the two coaxial ranges, O the centre, regarded as a point of the first range, O' its image, P, P' any pair of like points



then ∵ the ranges have no double point, [hyp.

∴ O lies between O', V' [th 32 cr.

Draw OS perpendicular to the axis and make it the geometric mean of O'O, OV'

then O'SV' is a right angle. [Geom.

Through S draw ISI' parallel to the axis, and so meeting it in the points at infinity I, I', of the two ranges the images of V, V';
then are the projective pencils S-IOVP, S-V'O'I'P' equal

For ∵ ∠ISV' = ∠OSO' = ∠VSI', [Geom.

∴ if the first pencil be turned through one of these angles, the three rays SI, SO SV coincide with SV', SO', SI',

∴ SP coincides with SP', [th 30 cor

∴ the two pencils are equal. Q.E.D.

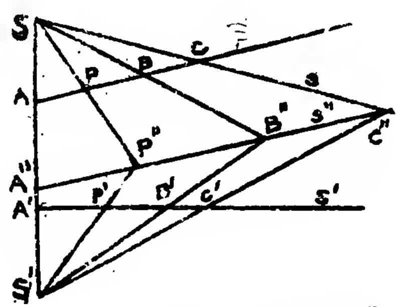
COR. *A pair of concentric projective pencils is a reflection of a pair of concentric equal pencils.*

It is convenient to speak of two coaxial projective ranges with two

double points, one, or none, as having two real and seperate double points, two real and coincident double points, or two imaginary double points, lying at the distances $\pm\sqrt{OV' \cdot OO'}$ from O, the centre of the ranges. When $OV' \cdot OO'$ is negative, the two points distant $\pm\sqrt{-OV' \cdot OO'}$ from O are the *ideals* of the double points.

So two concentric projective pencils have two real and seperate, two real and coincident, or two imaginary double rays.

PROB. 4. *Given three pairs of points / rays to construct two projective ranges. / pencils.*



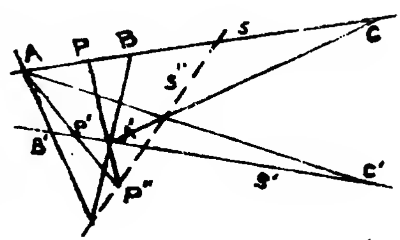
Let s, s' be two lines, A, B, C, three points on s, A', B', C', three like points on s';

1. On the co-line of any pair of like points, A, A', take S, S' any convenient centres, let B'' be the co-point of SB, S'B', C'' that of SC, S'C', and A'' that of AA', B''C'' then from S, A, B, C projects into A''B''C'' and from S', A''B''C'' projects into A', B', C'. Q.E.F. [th.30 cr3

Through S draw SI parallel to s cutting s'' in V'', then V' the co-point of S'V''-s', is the vanishing point of A'B'C'....

If the ranges be coaxial, from any convenient centre S project A, B, C into A'', B'', C''; then project A'', B'', C'' into A', B', C' as above.

2. Draw the coline s'', of AB'-A'B, AC'-A'C; draw A'P cutting s'' in P'', and AP'' cutting s' in P'.



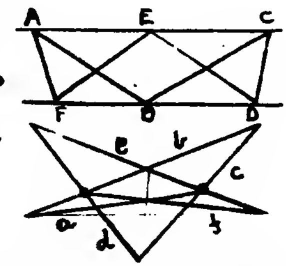
PROB. 5. *To find the double points of two coaxial projective ranges:*

construct the vanishing points V, V′, the centre O, its image O′; on O′V′ as diameter, draw a circle; then the circle whose centre is O and radius equal to the tangent from O to the circle O′V′ cuts the axis of the ranges in the points sought.

If O lie within the circle O′V′, then the circle whose centre is O and radius equal to the half chord perpendicular to O′V′ cuts the axis in the ideals of the imaginary double points.

EXAMPLES.

1. By methods here shown prove probs. 1, 2.



2. If the $\left\{\begin{matrix}\text{vertices}\\ \text{sides}\end{matrix}\right.$ of a hexagon, three and three $\left\{\begin{matrix}\text{lie on}\\ \text{pass thro}\end{matrix}\right.$ two $\left\{\begin{matrix}\text{lines,}\\ \text{points,}\end{matrix}\right.$ the three $\left\{\begin{matrix}\text{co-points}\\ \text{co-lines}\end{matrix}\right.$ of the three pairs of opposite $\left\{\begin{matrix}\text{sides}\\ \text{vertices}\end{matrix}\right.$ are $\left\{\begin{matrix}\text{colinear.}\\ \text{concurrent.}\end{matrix}\right.$

3. If one triangle circumscribe another, an infinite number of triangles may be drawn circumscribed about the first and inscribed in the other.

Through each vertex of the first triangle draw rays through the three vertices of the other; the pencils so formed are, two by two, in perspective, and any three like rays form one of the triangles sought.

§9. INVOLUTION.

If two $\left\{\begin{matrix}\text{coaxial ranges}\\ \text{concentric pencils}\end{matrix}\right.$ be projective figures and such that every $\left\{\begin{matrix}\text{point}\\ \text{ray}\end{matrix}\right.$ has the same image whether it be taken as an element of the one figure or of the other, they are $\left\{\begin{matrix}\textit{ranges}\\ \textit{pencils}\end{matrix}\right.$ *in involution*.

A $\left\{\begin{matrix}\text{point}\\ \text{ray}\end{matrix}\right.$ and its image are *conjugate* $\left\{\begin{matrix}\textit{points}\\ \textit{rays}\end{matrix}\right.$ of the involution.

THEOR. 36. *If in a pair of projective* $\left\{\begin{matrix}\textit{coaxial ranges}\\ \textit{concentric pencils}\end{matrix}\right.$ *a single* $\left\{\begin{matrix}\textit{point}\\ \textit{ray}\end{matrix}\right.$ *have the same image to whichever* $\left\{\begin{matrix}\textit{range}\\ \textit{pencil}\end{matrix}\right.$ *it belongs, the two* $\left\{\begin{matrix}\textit{ranges}\\ \textit{pencils}\end{matrix}\right.$ *are in involution.*

For, let $AA' \cdots PQ$, $A'A \cdots P'P$ be two coaxial projective ranges wherein the point A has the same image, A′, to whichever range it belongs;

then ∵ $(APQA') = (A'P'PA)$ [th. 22, cr. 3

$= (APP'A')$ [th. 18

∴ $Q \equiv P'$ [th. 20

i.e. every point has the same image to whichever range it belongs.

Q.E.D.

COR. 1. *Coaxial projective ranges whose vanishing points coincide, are in involution; and, conversely, in two coaxial projective ranges in involution the vanishing points coincide at the centre.*

COR. 2. *Two pairs of* $\left\{\begin{matrix}\textit{colinear points}\\ \textit{concurrent lines}\end{matrix}\right.$ *determine an involution.*

Let A, A′, B, B′ be two pairs of conjugate points in an involution, P any other point on the axis;

then ∵ any conjugate P′ of P satisfies the equation $(A'B'P'A) = (ABPA')$ [th. 22, cr. 3

and conversely, [th.

∴ P has one and but one such conjugate. Q.E.D.

THEOR. 37. *If* $\left\{\begin{matrix}A, A',\ B, B',\ C, C'\\ a, a',\ b, b',\ c, c'\end{matrix}\right.$ *be three pairs of conjugate* $\left\{\begin{matrix}\textit{points}\\ \textit{rays}\end{matrix}\right.$ *of an involution, then the product of the three cross-ratios* $\left\{\begin{matrix}(BAA'C), (CBB'A), (ACC'B)\\ (baa'c), (cbb'a), (acc'b)\end{matrix}\right.$ is -1; *and, conversely.*

For, $(BAA'C)\cdot(CBB'A)\cdot(ACC'B) = \overline{A'C\cdot B'A\cdot C'B} : \overline{BA'\cdot CB'\cdot AC'}$

$= -(AA'C'B):(A'ACB') = -1.$ Q.E.D. [th.22 cor.3

Conversely, if $(BAA'C)\cdot(CBB'A)\cdot(ACC'B) = -1$

then $(AA'C'B) = (A'ACB')$

and A, A', B, B', C, C' are conjugate points in involution. Q.E.D. [ths.30, 36.

COR. *If A, A', B, B', C, C' be three pairs of conjugate points of an involution, then $AB'\cdot BC'\cdot CA' + A'B\cdot B'C\cdot C'A = 0$; and conversely.*

Sometimes given as defn. of involu.

THEOR. 38. *If two points so slide upon an axis that the product of their distances from a fixed point upon it is constant, they generate two ranges in involution whose centre is the fixed point; and conversely, any two ranges in involution may be so generated.*

For $\because$ the ranges so found are projective ranges, wherein their vanishing points coincide at the fixed point, [th.32

$\therefore$ they are in involution. Q.E.D. [th.36 cor.1.

Conversely, if O be the centre of the involution;

then $\because$ $O \equiv V \equiv V'$, the vanishing points,

$\therefore$ the product $OP\cdot OP'$ is constant. Q.E.D. [th.32.

The constant product $OP\cdot OP'$ is the *power* of the involution.

COR. 1. *If the power of an involution be $\left\{\begin{array}{l}\text{positive}\\ \text{zero}\\ \text{negative}\end{array}\right.$ the involution has two $\left\{\begin{array}{l}\text{real and seperate}\\ \text{real and coincident}\\ \text{imaginary}\end{array}\right.$ double points,*

For, let O be the centre, S the double point, k^2 the power of the involution;

then $\because$ S is its own conjugate,

$\therefore OS^2 = k^2$

and there are two $\left\{\begin{array}{l}\text{real and seperate}\\ \text{real and coincident}\\ \text{imaginary}\end{array}\right.$ points S whose distances from O are $\pm\sqrt{k^2}$, when k^2 is $\left\{\begin{array}{l}\text{positive.}\\ \text{zero.}\\ \text{negative.}\end{array}\right.$

An involution that has two $\left\{\begin{array}{l}\text{real and seperate}\\ \text{real and coincident}\\ \text{imaginary.}\end{array}\right.$ double elements is a $\left\{\begin{array}{l}\text{positive}\\ \text{improper}\\ \text{negative}\end{array}\right.$ involution.

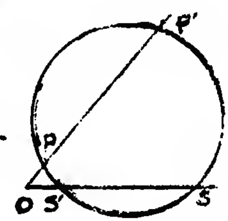
COR. 2. *In an improper involution the double element is the conjugate of every other element of the involution.*

THEOR. 39. *Circles through two fixed points cut any transversal in the conjugate points of an involution whose centre is the co-point of the transversal and the co-chord of the circles; and the involution is* $\left\{\begin{array}{l}\textit{positive}\\ \textit{negative}\end{array}\right.$ *when its centre is* $\left\{\begin{array}{l}\textit{without}\\ \textit{within}\end{array}\right.$ *the circles.*

For, let S, S′ be two fixed points, and let any transversal cut the chord SS′ and a circle through S, S′ in O, P, P′;

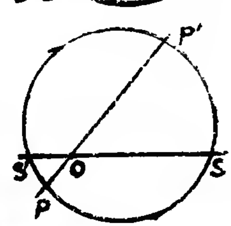
then $\because$ $OP\cdot OP' = OS\cdot OS' =$ a constant

$\therefore$ P, P′ are a pair of conjugate points of an involution whose centre is O.



and $\because$ $OS\cdot OS'$ is $\left\{\begin{array}{l}\text{positive}\\ \text{negative}\end{array}\right.$ when O is $\left\{\begin{array}{l}\text{without}\\ \text{within}\end{array}\right.$ the segment SS′,

$\therefore$ so is the involution. Q.E.D.



COR. *An involution is* $\left\{\begin{array}{l}\textit{positive}\\ \textit{negative}\end{array}\right.$ *when the segments of two pairs of conjugate elements* $\left\{\begin{array}{l}\textit{do not}\\ \textit{do}\end{array}\right.$ *overlap.*

THEOR. 40. *If a pair of variable* $\left\{\begin{matrix}\textit{points slide}\\ \textit{rays turn}\end{matrix}\right.$ *on* $\left\{\begin{matrix}\textit{an axis}\\ \textit{a point}\end{matrix}\right.$ *in harmonic ratio with two fixed* $\left\{\begin{matrix}\textit{points;}\\ \textit{rays}\end{matrix}\right.$ *they determine, an involution whose double* $\left\{\begin{matrix}\textit{points}\\ \textit{rays}\end{matrix}\right.$ *are the fixed* $\left\{\begin{matrix}\textit{points;}\\ \textit{rays;}\end{matrix}\right.$ *and, conversely, any positive involution may be so generated.*

For, the $\left\{\begin{matrix}\text{ranges}\\ \text{pencils}\end{matrix}\right.$ so determined are projective

and the fixed $\left\{\begin{matrix}\text{points}\\ \text{rays}\end{matrix}\right.$ are the double $\left\{\begin{matrix}\text{points}\\ \text{rays}\end{matrix}\right.$ [th. 33.

and ∵ any $\left\{\begin{matrix}\text{point}\\ \text{ray}\end{matrix}\right.$ taken as a $\left\{\begin{matrix}\text{point}\\ \text{ray}\end{matrix}\right.$ of either $\left\{\begin{matrix}\text{range}\\ \text{pencil}\end{matrix}\right.$ has the same conjugate [th. 25.

∴ the $\left\{\begin{matrix}\text{ranges}\\ \text{pencils}\end{matrix}\right.$ so generated are in involution. Q.E.D.

Conversely, if $\left\{\begin{matrix}S, T\\ s, t\end{matrix}\right.$ be the double $\left\{\begin{matrix}\text{points}\\ \text{rays}\end{matrix}\right.$ and $\left\{\begin{matrix}P, P'\\ p, p'\end{matrix}\right.$ be any pair of conjugate $\left\{\begin{matrix}\text{points}\\ \text{rays}\end{matrix}\right.$ of an involution;

then ∵ $\left\{\begin{matrix}P, P'\\ p, p'\end{matrix}\right.$ and $\left\{\begin{matrix}P', P\\ p', p\end{matrix}\right.$ cut $\left\{\begin{matrix}ST\\ st\end{matrix}\right.$ in the same cross-ratio, [th. 33.

∴ that cross-ratio is harmonic. Q.E.D.

COR. *Circles that cut a fixed circle at right angles, cut any diameter ST, of the fixed circle in an involution whose double points are S, T.* [th. 25 cr. 2.

THEOR. 41. *A pair of equal concentric pencils generated by a pair of rays at right angles to each other are in negative involution.*

For, the pencils are projective,

and $\because$ every ray has the same image, at right angles to it, to whichever pencil it $\because$ belongs,

and there is no double ray,

$\therefore$ the pencils are in negative involution. Q.E.D.

COR. *Every transversal cuts a pair of equal concentric pencils generated by a pair of rays at right angles to each other in a pair of coaxial ranges in negative involution.*

THEOR. 42. *Every pair of coaxial ranges in negative involution is a section of a pair of equal pencils generated by a pair of rays at right angles to each other.*

For, let $-k^2$ be the power of an involution, O its centre, P, P′ any pair of conjugate points;

then $\because$ $OP.OP' = -k^2$

$\therefore$ O lies between P, P′.

Draw OS perpendicular to PP′ and take S such that $OS = k$;

then $\because$ OS is the geometric mean of PO, OP′

$\therefore$ PSP′ is a right angle. Q.E.D. [Geom.

COR. *Every pair of pencils in involution is a reflection of a pair of equal pencils whose like rays are at right angles.*

THEOR. 43. *If two pencils be in involution there is one pair of conjugate rays at right angles; and if there be two such pairs, then every pair is at right angles.*

For, let S-ab⋯, S-a′b′⋯ be two pencils in involution, and let a transversal s cut these pencils in AB⋯, A′B′⋯, two ranges in involution.

Draw the circle SAB, SA′B′ and let T be their second co-point.

Through S, T and with centre on s, draw a circle cutting s in G, G′;

then ∵ G, G′ is a diameter of the circle and S a point upon it,

∴ the rays g g′ are at right angles;

and ∵ G, G′ are a pair of conjugate points [th 34 or 2, th 39

∴ g, g′ are a pair of conjugate rays of the involution. Q.E.D.

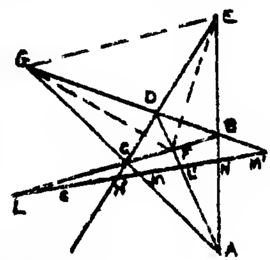
And if another pair of conjugate rays be at right angles,

then a second point on s is equally distant from S, T,

∴ s is perpendicular to ST at its middle point, and every point of s is equally distant from S, T, [Geom.

∴ every pair of conjugate rays are at right angles to each other. Q.E.D.

THEOR. 44. *The three pairs of opposite sides/vertices of a four-point/four-line cut/project any transversal/centre in three pairs of conjugate points/rays of an involution.*



For, let ABCD be any four-point, EFG its diagonal triangle, s any transversal that cuts the opposite sides BC, AD in L, L′, the sides CA, BD in M, M′, the sides AB, CD in N, N′; from C, D project the ranges MEBA, ENAB into the ranges M′LMN, NN′LM′ all projective

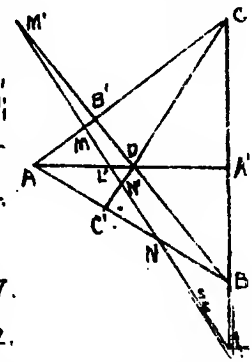
then ∵ N has the same image to whichever range it belongs

∴ L, L′, M, M′, N, N′ are pairs of conjugate points of an involution. [th 36

Q.E.D.

THEOR. 45. *If the three sides of a triangle be cut by a transversal and by concurrent lines through the vertices, the product of the three cross-ratios so formed is −1; and, conversely, if the sides of a triangle be so divided by a transversal/three concurrent lines thro the vertices and three points, then these points lie on concurrent lines thro the vertices/are colinear.*

For, let ABC be any triangle, let BC, CA, AB be cut by a transversal s in L, M, N and by lines through a point D and the vertices in A', B', C', and let s be cut by these lines in L', M', N';



then ∵ L, L', M, M', N, N' are pairs of conjugate points in involution, [th. 44.

∴ the pencil $D\text{-}LL'MM'NN'$ is in involution

∴ $D\text{-}(M'LL'N') \cdot D\text{-}(N'MM'L') \cdot D\text{-}(L'NN'M') = -1$ [th. 37.

∴ $(BLA'C) \cdot (CMB'A) \cdot (ANC'B) = -1$ [th. 22.

Conversely. Let $(BLA'C) \cdot (CMB'A) \cdot (ANC'B) = -1$, and let L, M, N be colinear; let D be the copoint of BB', CC', and from D project A into A'';

then ∵ $(BLA''C) \cdot (CMB'A) \cdot (ANC'B) = -1$ [above

∴ $(BLA''C) = (BLA'C)$

∴ $A'' \equiv A'$. Q.E.D.

COR. Ceva's theorem. *Concurrent lines thro the vertices of a triangle*
Menelaus' theorem. *Colinear points upon the sides of a triangle*

angle divide the sides in ratios whose product is $\begin{cases}+1;\\-1;\end{cases}$ *and, conversely.*

For ∵ $(BLA'C) \cdot (CMB'A) \cdot (ANC'B) = -1$ [th.

and ∵ this product is the ratio of the product of the ratios of division by the concurrent lines to the product of the ratios of division by the colinear points,

and ∵ the position of the colinear points is independent of that of the concurrent lines.

∴ the two products are seperately constant.

and ∵ if the points be taken on the line at infinity the product of their ratios of division is -1

∴ that product is always -1, and the other is $+1$. Q.E.D.

omit If the parameter of an homology be -1, the homology is *harmonic*.

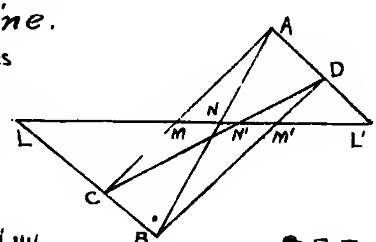
THEOR. 46. *If two figures be in harmonic homology, every* $\left\{\begin{matrix}\textit{point}\\ \textit{line}\end{matrix}\right.$ *has the same image to whichever figure it belongs; and, conversely, if of two figures in homology one* $\left\{\begin{matrix}\textit{point}\\ \textit{line}\end{matrix}\right.$ *has the same image to whichever figure it belongs the two figures are in harmonic homology.*

For a point and its image are colinear with the centre and are harmonic conjugates as to the centre and axis.

Conversely. A single point and its image determine the parameter of the homology.

COR. *In harmonic homology the vanishing lines coincide midway between the centre and axis.*

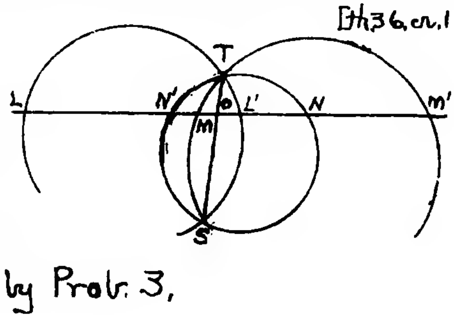
PROB. 6. *Given two pairs of conjugate* $\left\{\begin{matrix}\textit{points}\\ \textit{lines}\end{matrix}\right.$ *of an involution, to construct the conjugate of a fifth* $\left\{\begin{matrix}\textit{point.}\\ \textit{line.}\end{matrix}\right.$



1. Let L, L', M, M' be two pairs of conjugate points of an involution and N a fifth point; through N draw any convenient line AB, and let C, D be the co-points of AM-BL, AL'-BM';
then is N', the co-point of LM-CD, the point sought. [th. 44.

Q.E.F.

If N be the point at infinity of LM, then AB is parallel to LM and N' is the centre of the involution. [th. 36, cr. 1.



2. (for ranges only.) Through LL', MM', as chords, draw convenient circles meeting in S, T then the circle STN cuts the axis in N', the point sought. [th. 39.

The double points may be found by Prob. 3.

EXAMPLES.

1. If three $\left\{\begin{matrix}\text{points}\\\text{rays}\end{matrix}\right.$ of an involution $\left\{\begin{matrix}\text{lie upon}\\\text{pass thro}\end{matrix}\right.$ the three $\left\{\begin{matrix}\text{sides}\\\text{vertices}\end{matrix}\right.$ of a triangle the $\left\{\begin{matrix}\text{co-lines}\\\text{co-points}\end{matrix}\right.$ of their conjugates with the opposite $\left\{\begin{matrix}\text{vertices}\\\text{sides}\end{matrix}\right.$ of the triangle are $\left\{\begin{matrix}\text{concurrent.}\\\text{colinear.}\end{matrix}\right.$

2. The three $\left\{\begin{matrix}\text{points}\\\text{lines}\end{matrix}\right.$ that, with a given $\left\{\begin{matrix}\text{line,}\\\text{point,}\end{matrix}\right.$ divide harmonically the $\left\{\begin{matrix}\text{segments}\\\text{angles}\end{matrix}\right.$ between the opposite $\left\{\begin{matrix}\text{vertices}\\\text{sides}\end{matrix}\right.$ of a $\left\{\begin{matrix}\text{four-line}\\\text{four-point}\end{matrix}\right.$ are $\left\{\begin{matrix}\text{colinear.}\\\text{concurrent.}\end{matrix}\right.$

3. The mid-points of the diagonals of a four-line are colinear.

4. A pair of $\left\{\begin{matrix}\text{coaxial}\\\text{concentric}\end{matrix}\right.$ opposite $\left\{\begin{matrix}\text{ranges}\\\text{pencils}\end{matrix}\right.$ are in positive involution.

5. If the sides of a triangle be cut at equal angles by rays through a point on the circumscribing circle, the section points are colinear.

6. If the segments between the three vertices of a triangle and three points on a tangent to the inscribed circle subtend constant angles at the centre of the circle, the three segments are concurrent.

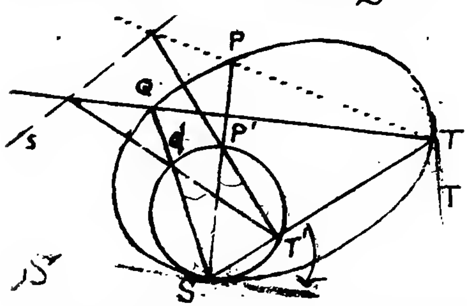
7. Prove directly from elementary geometry the theorems of Menelaus and Ceva.

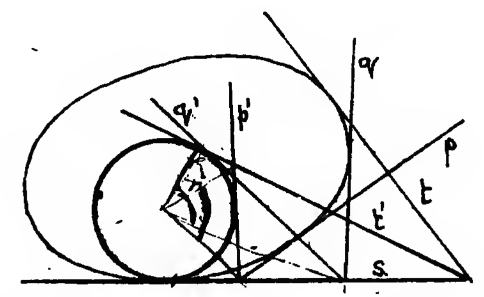
8. The co-lines of the vertices of a triangle with the points ~~of~~ contact of the opposite sides and the inscribed circle are co~~linear~~ncurrent.

9. If in two coplanar projective figures a $\left\{\begin{matrix}\text{point}\\\text{line}\end{matrix}\right.$ have the same image to whichever figure it belongs, then every $\left\{\begin{matrix}\text{point}\\\text{line}\end{matrix}\right.$ has the same image.

S10. GENERAL PROPERTIES OF CONICS.

THEOR. 47. *If two* $\left\{\begin{array}{l}\textit{pencils}\\ \textit{ranges}\end{array}\right.$ *be projective, the* $\left\{\begin{array}{l}\textit{locus}\\ \textit{envelop}\end{array}\right.$ *of the* $\left\{\begin{array}{l}\textit{co-points}\\ \textit{co-lines}\end{array}\right.$ *of like* $\left\{\begin{array}{l}\textit{rays}\\ \textit{points,}\end{array}\right.$ *is a conic* $\left\{\begin{array}{l}\textit{thro}\\ \textit{touching}\end{array}\right.$ *the* $\left\{\begin{array}{l}\textit{centres}\\ \textit{axes}\end{array}\right.$ *of the* $\left\{\begin{array}{l}\textit{pencils;}\\ \textit{ranges;}\end{array}\right.$ *and, conversely, the* $\left\{\begin{array}{l}\textit{co-lines}\\ \textit{co-points}\end{array}\right.$ *of a variable* $\left\{\begin{array}{l}\textit{point upon}\\ \textit{tangent to}\end{array}\right.$ *a conic with two fixed* $\left\{\begin{array}{l}\textit{points of}\\ \textit{tangents to}\end{array}\right.$ *the conic, generate a pair of projective* $\left\{\begin{array}{l}\textit{pencils.}\\ \textit{ranges.}\end{array}\right.$





For, let $\left\{\begin{array}{l}S\text{-}STPQ\cdots,\ T\text{-}STPQ\cdots\\ s\text{-}stpq\cdots,\ t\text{-}stpq\cdots\end{array}\right.$ be two projective $\left\{\begin{array}{l}\text{pencils,}\\ \text{ranges,}\end{array}\right.$ wherein $\left\{\begin{array}{l}SS\\ ss\end{array}\right.$ is the image in the first $\left\{\begin{array}{l}\text{pencil}\\ \text{range}\end{array}\right.$ of $\left\{\begin{array}{l}TS\\ ts\end{array}\right.$ in the second; through $\left\{\begin{array}{l}S\\ ss\end{array}\right.$ and tangent to $\left\{\begin{array}{l}SS\\ s\end{array}\right.$ draw a circle O, and let $\left\{\begin{array}{l}T', P', Q', \cdots\\ t', p', q', \cdots\end{array}\right.$ be the $\left\{\begin{array}{l}\text{co-points of}\\ \text{co-tangents to}\end{array}\right.$ this circle and the $\left\{\begin{array}{l}\text{rays } ST, SP, SQ, \cdots;\\ \text{points } st, sp, sq, \cdots;\end{array}\right.$

then $\because$ every pair of $\left\{\begin{array}{l}\text{rays, } SP', T'P', \cdots \text{ meet at}\\ \text{points, } sp', t'p', \cdots \text{ subtend at O}\end{array}\right.$ a constant angle, [measd by half arc $\left\{\begin{array}{l}ST'\\ st'\end{array}\right.$

$\therefore$ the $\left\{\begin{array}{l}\text{pencils } S\text{-}SP'Q'\cdots,\ T'\text{-}SP'Q'\cdots\\ \text{ranges } s\text{-}sp'q'\cdots,\ t'\text{-}sp'q'\cdots\end{array}\right.$ are projective,

$\therefore$ the $\left\{\begin{array}{l}\text{pencils } T'\text{-}SP'Q'\cdots,\ T\text{-}SPQ\cdots\\ \text{ranges } t'\text{-}sp'q'\cdots,\ t\text{-}spq\cdots\end{array}\right.$ are projective;

and $\because \left\{\begin{array}{l} TS \equiv T'S \\ ts \equiv t's \end{array}\right.$

$\therefore$ the $\left\{\begin{array}{l} \text{pencils } T'\text{-}SP'Q'\cdots,\ T\text{-}SPQ\cdots \\ \text{ranges } t'\text{-}sp'q'\cdots,\ t\text{-}spq\cdots \end{array}\right.$ are in perspective,

and $\left\{\begin{array}{l} \text{reflect} \\ \text{project} \end{array}\right.$ into each other from a fixed $\left\{\begin{array}{l} \text{axis;} \\ \text{centre;} \end{array}\right.$ [th. 30. cor. 2

$\therefore$ the figure $\left\{\begin{array}{l} STPQ\cdots \\ stpq\cdots \end{array}\right.$ is in perspective with the circle $\left\{\begin{array}{l} ST'P'Q'\cdots, \\ st'p'q'\cdots, \end{array}\right.$

and is a conic. Q.E.D. [th. " df. conic.

Conversely, let $\left\{\begin{array}{l} S'T'P'Q'\cdots \\ s't'p'q'\cdots \end{array}\right.$ be any circular image of a conic $\left\{\begin{array}{l} STPQ\cdots; \\ stpq\cdots; \end{array}\right.$

then $\because$ the $\left\{\begin{array}{l} \text{pencils } S'\text{-}P'Q'\cdots,\ T'\text{-}P'Q'\cdots \\ \text{ranges } s'\text{-}p'q'\cdots,\ t'\text{-}p'q'\cdots \end{array}\right.$ are projective with $\left\{\begin{array}{l} S\text{-}PQ\cdots,\ T\text{-}PQ\cdots; \\ s\text{-}pq\cdots,\ t\text{-}pq\cdots; \end{array}\right.$

and with each other, [hyp. geom.

$\therefore \left\{\begin{array}{l} S\text{-}PQ\cdots,\ T\text{-}PQ\cdots \\ s\text{-}pq\cdots,\ t\text{-}pq\cdots \end{array}\right.$ are projective. Q.E.D.

If two $\left\{\begin{array}{l} \text{pencils} \\ \text{ranges} \end{array}\right.$ be in perspective they determine an *improper conic*, consisting of two $\left\{\begin{array}{l} \text{lines,} \\ \text{points,} \end{array}\right.$ the $\left\{\begin{array}{l} \text{axis} \\ \text{centre} \end{array}\right.$ of perspective and the $\left\{\begin{array}{l} \text{co-ray} \\ \text{co-point} \end{array}\right.$ of the $\left\{\begin{array}{l} \text{pencils.} \\ \text{ranges.} \end{array}\right.$

COR. 1. *The images of the* $\left\{\begin{array}{l} \textit{co-rays} \\ \textit{co-points} \end{array}\right.$ *of the two* $\left\{\begin{array}{l} \textit{pencils} \\ \textit{ranges} \end{array}\right.$ *are the* $\left\{\begin{array}{l} \textit{lines of tangency at the centres.} \\ \textit{points of contact of the axes.} \end{array}\right.$

For, let $\left\{\begin{array}{l} P \\ p \end{array}\right.$ approach the limiting position $\left\{\begin{array}{l} S, \\ s, \end{array}\right.$

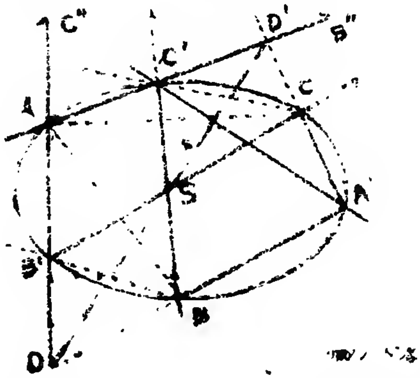
then $\because \left\{\begin{array}{l} SP \\ sp \end{array}\right.$ approaches the image of $\left\{\begin{array}{l} TS \\ ts \end{array}\right.$ i.e. to $\left\{\begin{array}{l} SS \\ ss \end{array}\right.$

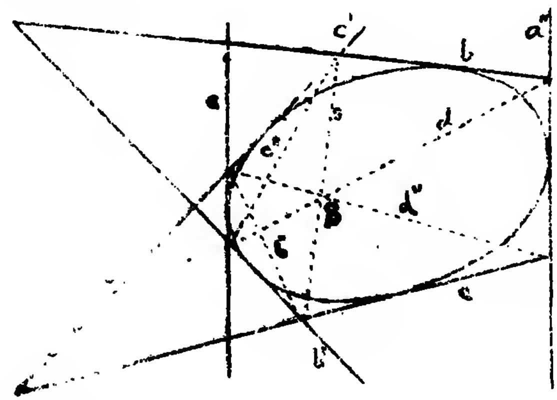
$\therefore \left\{\begin{matrix} SS \\ ss \end{matrix}\right.$ is the $\left\{\begin{matrix} \text{tangent at } S, \\ \text{contact of } s, \end{matrix}\right.$ and $\left\{\begin{matrix} TT \\ tt \end{matrix}\right.$ is the $\left\{\begin{matrix} \text{tangent at } T. \\ \text{contact at } t. \end{matrix}\right.$ Q.E.D.

Cor. 2. One conic and but one can $\left\{\begin{matrix} \textit{pass thro} \\ \textit{touch} \end{matrix}\right.$ *five given* $\left\{\begin{matrix} \textit{points} \\ \textit{lines} \end{matrix}\right.$ *no four of which are* $\left\{\begin{matrix} \textit{co-linear;} \\ \textit{concurrent.} \end{matrix}\right.$

2nd inverted

THEOR. 48 $\left[\begin{matrix} \text{Pascal's theorem.} \\ \text{Brianchon's theorem.} \end{matrix}\right]$ *In a hexagon whose* $\left\{\begin{matrix} \textit{vertices lie on} \\ \textit{sides touch} \end{matrix}\right.$ *a conic, the three pairs of opposite* $\left\{\begin{matrix} \textit{sides reflect} \\ \textit{vertices project} \end{matrix}\right.$ *into each other from a fixed* $\left\{\begin{matrix} \textit{axis;} \\ \textit{centre;} \end{matrix}\right.$ *and, conversely,*

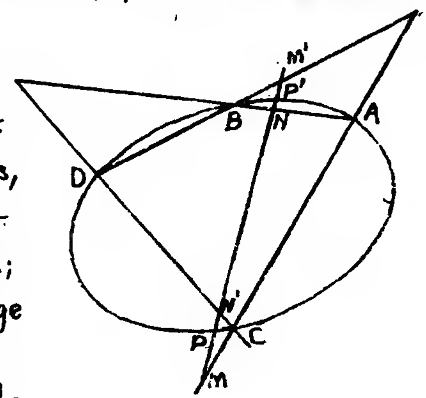




For, let AB'CA'BC' lie a hexagon inscribed in a conic;
then ∵ the pencils of rays drawn from B, C as centres to the other vertices are projective, and the two rays to any same point are images, [th. 47
∴ AB', AC' cut the pencils B, C, in ranges AB'DC'', AB[illegible] that are in perspective, [th. 30. cr. 3.
wherein, BC'-B'C is the centre of perspective
and AB'-A'B, CA'-C'A are images D, D',
∴ AB'-A'B, BC'-B'C, CA'-C'A are colinear. Q.E.D.

THEOR. 49. [De Sargues' theorem] *The* $\left\{\begin{matrix}\textit{co-points of}\\ \textit{tangents from}\end{matrix}\right.$ *any* $\left\{\begin{matrix}\textit{line and}\\ \textit{point to}\end{matrix}\right.$ *a conic are conjugate* $\left\{\begin{matrix}\textit{points}\\ \textit{rays}\end{matrix}\right.$ *of the involution determined by any* $\left\{\begin{matrix}\textit{inscribed four-point.}\\ \textit{circumscribed four-line.}\end{matrix}\right.$

For, let ABCD be any four-point inscribed in a conic and let a line s cut the pairs of opposite sides AC, BD, BC, AD in M, M′, N, N′ and cut the conic in P, P′



then M, M′, N, N′ determine the involution given by ABCD on s [th. 44 cr.

and ∵ A-PP′BC, D-P′PCB are projective pencils, [ths. 47, 14.

∴ PP′NM, P′PN′M′ are projective ranges;

and ∵ P has P′ for image to whichever range it belongs,

∴ P, P′ are a pair of conjugate points of the involution MM′NN′. Q.E.D.

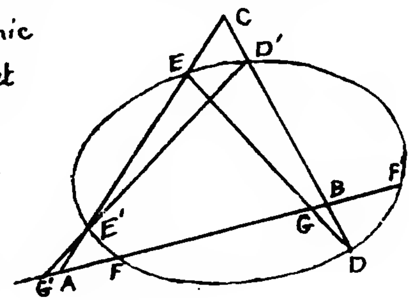
A system of conics that $\left\{\begin{matrix}\text{pass thro}\\ \text{touch}\end{matrix}\right.$ four $\left\{\begin{matrix}\text{points}\\ \text{lines}\end{matrix}\right.$ is a $\left\{\begin{matrix}\textit{pencil}\\ \textit{range}\end{matrix}\right.$ of conics.

COR. *If a* $\left\{\begin{matrix}\textit{pencil}\\ \textit{range}\end{matrix}\right.$ *of conics be* $\left\{\begin{matrix}\textit{cut}\\ \textit{touched}\end{matrix}\right.$ *by any* $\left\{\begin{matrix}\textit{transversal}\\ \textit{concentric rays}\end{matrix}\right.$ *the pairs of* $\left\{\begin{matrix}\textit{points}\\ \textit{tangents}\end{matrix}\right.$ *are pairs of conjugate* $\left\{\begin{matrix}\textit{points}\\ \textit{rays}\end{matrix}\right.$ *in involution; and if the* $\left\{\begin{matrix}\textit{transversal be tangent to}\\ \textit{center lie upon}\end{matrix}\right.$ *one of the conics, the* $\left\{\begin{matrix}\textit{contact}\\ \textit{tangent}\end{matrix}\right.$ *is a double* $\left\{\begin{matrix}\textit{point}\\ \textit{ray}\end{matrix}\right.$ *of the involution.*

Omit

THEOR. 50. [Carnot's theorem] *The six* $\left\{\begin{matrix}\textit{points of}\\ \textit{tangents to}\end{matrix}\right.$ *a conic that* $\left\{\begin{matrix}\textit{lie,}\\ \textit{pass,}\end{matrix}\right.$

two and two $\left\{\begin{array}{l}\textit{upon}\\ \textit{thro}\end{array}\right.$ *the* $\left\{\begin{array}{l}\textit{sides}\\ \textit{vertices}\end{array}\right.$ *of a triangle, divide the sides in six ratios of division whose product is unity; and, conversely.*



For, let ABC be a triangle, and let a conic cut BC in D, D', CA in E, E', AB in F, F', and let DE, D'E' cut AB in G, G';

then ∵ A, B, F, F', G, G' are conjugate points of an involution given by the four-point DD'EE' [th. 49

$$\therefore\ (AFGB) = (BF'G'A) = (AG'F'B)$$

i.e. $$\frac{AG}{GB}\cdot\frac{AG'}{G'B} = \frac{AF}{FB}\cdot\frac{AF'}{F'B};$$

and ∵ the points D, E, G, D', E', G' so divide the sides that

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AG}{GB}\times\frac{BD'}{D'C}\cdot\frac{CE'}{E'A}\cdot\frac{AG'}{G'B} = -1\times-1 = 1$$ [th. 45. cr.

$$\therefore\ \frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}\times\frac{BD'}{D'C}\cdot\frac{CE'}{E'A}\cdot\frac{AF'}{F'B} = 1$$

Conversely, let D, D', E, E', F, F' be points on the sides of a triangle BC, CA, AB such that $\frac{BD}{DC}\cdot\frac{BD'}{D'C}\cdot\frac{CE}{EA}\cdot\frac{CE'}{E'A}\cdot\frac{AF}{FB}\cdot\frac{AF'}{F'B} = 1$;

through D, D', E, E', F pass a conic whose second copoint with AB is F''

then ∵ $$\frac{BD}{DC}\cdot\frac{BD'}{D'C}\cdot\frac{CE}{EA}\cdot\frac{CE'}{E'A}\cdot\frac{AF}{FB}\cdot\frac{AF''}{F''B} = 1$$ [above

$$\therefore\ \frac{AF'}{F'B} = \frac{AF''}{F''B} \text{ and } F'' \equiv F'$$ Q.E.D.

PROB. 7. *To construct* $\left\{\begin{array}{l}\textit{points of}\\ \textit{tangents to}\end{array}\right.$ *the conic given by five* $\left\{\begin{array}{l}\textit{points:}\\ \textit{tangents:}\end{array}\right.$

1. With two of the given $\left\{\begin{array}{l}\text{points}\\ \text{tangents}\end{array}\right.$ as $\left\{\begin{array}{l}\text{centres}\\ \text{axes}\end{array}\right.$ construct the projective

$\left\{\begin{array}{l}\text{pencils}\\\text{ranges}\end{array}\right.$ given by the other three, the $\left\{\begin{array}{l}\text{co-points}\\\text{co-lines}\end{array}\right.$ of $\left\{\begin{array}{l}\text{rays}\\\text{points}\end{array}\right.$ with their images are the $\left\{\begin{array}{l}\text{points}\\\text{tangents}\end{array}\right.$ sought.

2. $\left\{\begin{array}{l}\text{Thro}\\\text{Upon}\end{array}\right.$ the first of the given $\left\{\begin{array}{l}\text{points } A, B', C, A', B\\\text{tangents } a, b', c, a', b\end{array}\right.$ take any $\left\{\begin{array}{l}\text{line}\\\text{point}\end{array}\right.$ and $\left\{\begin{array}{l}\text{upon}\\\text{thro}\end{array}\right.$ it find the $\left\{\begin{array}{l}\text{vertex } C'\\\text{side } c'\end{array}\right.$ of the hexagon as in the converse to $\left\{\begin{array}{l}\text{Pascal's}\\\text{Brianchon's}\end{array}\right.$ theorem.

3. $\left\{\begin{array}{l}\text{Thro}\\\text{Upon}\end{array}\right.$ one of the given $\left\{\begin{array}{l}\text{points, } A,\\\text{tangents, } a,\end{array}\right.$ take any $\left\{\begin{array}{l}\text{line,}\\\text{point,}\end{array}\right.$ the opposite $\left\{\begin{array}{l}\text{sides}\\\text{vertices}\end{array}\right.$ of the $\left\{\begin{array}{l}\text{four-point}\\\text{four-line}\end{array}\right.$ given by the other four $\left\{\begin{array}{l}\text{points}\\\text{tangents}\end{array}\right.$ give an involution $\left\{\begin{array}{l}\text{upon}\\\text{at}\end{array}\right.$ this $\left\{\begin{array}{l}\text{line,}\\\text{point,}\end{array}\right.$ and the conjugate of $\left\{\begin{array}{l}A\\a\end{array}\right.$ is a $\left\{\begin{array}{l}\text{point of}\\\text{tangent to}\end{array}\right.$ the conic.

NOTE. These constructions may be made when two of the given $\left\{\begin{array}{l}\text{points}\\\text{tangents}\end{array}\right.$ coincide if their $\left\{\begin{array}{l}\text{co-line}\\\text{co-point}\end{array}\right.$ be given, a $\left\{\begin{array}{l}\text{point}\\\text{tangent}\end{array}\right.$ and the $\left\{\begin{array}{l}\text{tangent at}\\\text{contact of}\end{array}\right.$ it counting for two $\left\{\begin{array}{l}\text{points.}\\\text{tangents.}\end{array}\right.$

PROB. 8. *Given five* $\left\{\begin{array}{l}\textit{points of}\\\textit{tangents to}\end{array}\right.$ *a conic, to construct the* $\left\{\begin{array}{l}\textit{tangents at}\\\textit{contacts of}\end{array}\right.$ *them.*

1. Draw the $\left\{\begin{array}{l}\text{Pascal line } s\\\text{Brian. point } S\end{array}\right.$ of the hexagon $\left\{\begin{array}{l}AABCA'B\\aabca'b\end{array}\right.$ and construct $\left\{\begin{array}{l}s\text{-}CA'\\S\text{-}ca'\end{array}\right.$ as a second $\left\{\begin{array}{l}\text{point of}\\\text{line thro}\end{array}\right.$ the $\left\{\begin{array}{l}\text{tangent } AA.\\\text{contact } aa.\end{array}\right.$ [Th. 48

2. In the involution formed $\left\{\begin{array}{l}\text{on } BB'\\\text{at } bb'\end{array}\right.$ by the opposite $\left\{\begin{array}{l}\text{sides}\\\text{vertices}\end{array}\right.$ of the

$\left\{\begin{array}{l}\text{four-point } AAA'C\\ \text{four-line } aaa'c\end{array}\right.$ construct the conjugate of the $\left\{\begin{array}{l}\text{point } A'C\text{-}BB'\\ \text{line } a'c\text{-}bb'\end{array}\right.$ as a second $\left\{\begin{array}{l}\text{point on } AA'\\ \text{line thro } aa'\end{array}\right.$ [th. 49.

PROB. 9. *Given four* $\left\{\begin{array}{l}\textit{points}\\ \textit{tangents}\end{array}\right.$ *and a* $\left\{\begin{array}{l}\textit{tangent}\\ \textit{point}\end{array}\right.$ *to construct the conic.*

In the involution formed $\left\{\begin{array}{l}\text{on } b\\ \text{at } B\end{array}\right.$ by the opposite $\left\{\begin{array}{l}\text{sides}\\ \text{vertices}\end{array}\right.$ of the $\left\{\begin{array}{l}\text{four-point}\\ \text{four-line}\end{array}\right.$ $\begin{array}{l}AB'CA'\\ ab'ca'\end{array}$ construct a double $\left\{\begin{array}{l}\text{point } B\\ \text{ray } b\end{array}\right.$ the $\left\{\begin{array}{l}\text{contact of } b;\\ \text{tangent at } B;\end{array}\right.$ then as in Pr. 8 [th. 49. cr.

If the involution be positive two conics are possible; if negative, none.

PROB. 10. *Given three* $\left\{\begin{array}{l}\textit{points } A, B, C\\ \textit{tangents } a, b, c\end{array}\right.$ *and two* $\left\{\begin{array}{l}\textit{tangents } p, q,\\ \textit{points } P, Q,\end{array}\right.$ *to construct the conic.*

Let $\left\{\begin{array}{l}P, Q\\ p, q\end{array}\right.$ be the $\left\{\begin{array}{l}\text{contacts of } p, q\\ \text{tangents at } P, Q\end{array}\right.$ and construct the involutions determined by the $\left\{\begin{array}{l}\text{inscribed four-point } PPQQ\\ \text{circumscribed four-line } ppqq\end{array}\right.$ upon $\left\{\begin{array}{l}AB, AC;\\ ab, ac;\end{array}\right.$ in these involutions take the $\left\{\begin{array}{l}\text{co-line}\\ \text{co-point}\end{array}\right.$ of two double $\left\{\begin{array}{l}\text{points,}\\ \text{rays,}\end{array}\right.$ one from each, as the $\left\{\begin{array}{l}\text{co-line of contacts } P, Q,\\ \text{co-point of tangents } p, q,\end{array}\right.$ and thence, with the five $\left\{\begin{array}{l}\text{points}\\ \text{tangents}\end{array}\right.$ so known, as in Prs. 6, 7.

For $\left\{\begin{array}{l}PQ\\ pq\end{array}\right.$ is a self-opposite $\left\{\begin{array}{l}\text{side}\\ \text{vertex}\end{array}\right.$ of the $\left\{\begin{array}{l}\text{four-point } PPQQ\\ \text{four-line } ppqq\end{array}\right.$ and determines a coincident pair of conjugates.

If both involutions be positive there are four conics; but if of the three given $\left\{\begin{array}{l}\text{points}\\ \text{tangents}\end{array}\right.$ the $\left\{\begin{array}{l}\text{co-line}\\ \text{co-point}\end{array}\right.$ of two be $\left\{\begin{array}{l}\text{concurrent}\\ \text{colinear}\end{array}\right.$ with the two remaining

elements, these four conics reduce to two, and to a pair of $\left\{\begin{matrix}\text{lines}\\ \text{points}\end{matrix}\right.$ taken twice.

A circle may be counted a conic that passes through two fixed points, the two imaginary *circular* points at infinity.

E.g. Three points determine a circle; two points and a tangent, two circles; a point and two tangents, or three tangents, four circles. Circles through two points determine an involution upon any transversal.

PROB. 11. *To find the* $\left\{\begin{matrix}\textit{co-points}\\ \textit{co-lines}\end{matrix}\right.$ *of a conic* $\left\{\begin{matrix}\text{STABC}\\ \text{stabc}\end{matrix}\right.$ *and a* $\left\{\begin{matrix}\textit{line s:}\\ \textit{point S:}\end{matrix}\right.$

Find the double points of the ranges in which the pencils S-ABC, T-ABC are cut by s. There are always two such points, real or imaginary.

This problem includes the determination of $\left\{\begin{matrix}\text{the directions to infinity of the conic.}\\ \text{the tangents parallel to a given line.}\end{matrix}\right.$

PROB. 12. *Given two* $\left\{\begin{matrix}\textit{co-points}\\ \textit{co-tangents}\end{matrix}\right.$ *of two conics, to find the other* $\left\{\begin{matrix}\textit{co-points:}\\ \textit{co-tangents:}\end{matrix}\right.$

Let ABLMN, ABPQR be the two conics, and in ABPQR construct a four-point ABL′M′ such that AL′ ≡ AL, BM′ ≡ BM, and let H ≡ LM-L′M′. So, with the four-point ABMN, find another point K; then HK cuts either conic in the points sought. [Th. 49.

Two conics *touch* each other when two of their $\left\{\begin{matrix}\text{co-points}\\ \text{co-tangents}\end{matrix}\right.$ coincide; if they touch again they have *double contact*; if three $\left\{\begin{matrix}\text{co-points}\\ \text{co-tangents}\end{matrix}\right.$ coincide, the two conics *osculate*.

EXAMPLES.

1. $\left[\begin{matrix}\text{"Newton's}\\\text{Correlative of}\end{matrix}\right.$ organic description of a conic.") If a pair of constant $\left\{\begin{matrix}\text{angles}\\\text{segments}\end{matrix}\right.$ so $\left\{\begin{matrix}\text{turn}\\\text{slide}\end{matrix}\right.$ upon their $\left\{\begin{matrix}\text{vertices}\\\text{sides}\end{matrix}\right.$ that the $\left\{\begin{matrix}\text{co-point}\\\text{co-line}\end{matrix}\right.$ of one pair of $\left\{\begin{matrix}\text{sides,}\\\text{ends,}\end{matrix}\right.$ one from each $\left\{\begin{matrix}\text{angle, trace}\\\text{segment, envelop}\end{matrix}\right.$ a $\left\{\begin{matrix}\text{line,}\\\text{point,}\end{matrix}\right.$ the $\left\{\begin{matrix}\text{co-point}\\\text{co-line}\end{matrix}\right.$ of the other pair $\left\{\begin{matrix}\text{traces}\\\text{envelops}\end{matrix}\right.$ a conic.

2. If two fixed tangents to a parabola be cut by a variable tangent, the ranges so found are similar; and, conversely, the envelop of the co-lines of like points of similar ranges is a parabola.

3. The co-points of like rays of two opposite pencils lie on an hyperbola whose asymptotes are at right angles.

Such an hyperbola is a *rectangular hyperbola.*

4. If two triangles be in homology, the $\left\{\begin{matrix}\text{co-points}\\\text{co-lines}\end{matrix}\right.$ of the $\left\{\begin{matrix}\text{sides}\\\text{vertices}\end{matrix}\right.$ of the one with the unlike $\left\{\begin{matrix}\text{sides}\\\text{vertices}\end{matrix}\right.$ of the other $\left\{\begin{matrix}\text{lie on}\\\text{touch}\end{matrix}\right.$ a conic.

5. If the three $\left\{\begin{matrix}\text{sides}\\\text{vertices}\end{matrix}\right.$ of a triangle $\left\{\begin{matrix}\text{turn}\\\text{slide}\end{matrix}\right.$ on three fixed $\left\{\begin{matrix}\text{points}\\\text{lines}\end{matrix}\right.$ while two $\left\{\begin{matrix}\text{vertices slide}\\\text{sides turn}\end{matrix}\right.$ on two fixed $\left\{\begin{matrix}\text{lines}\\\text{points}\end{matrix}\right.$ the third $\left\{\begin{matrix}\text{vertex traces}\\\text{side envelops}\end{matrix}\right.$ a conic.

6. If a conic $\left\{\begin{matrix}\text{circumscribe a four-point}\\\text{envelop a four-line}\end{matrix}\right.$ the rectangle of the distances from any $\left\{\begin{matrix}\text{point}\\\text{tangent}\end{matrix}\right.$ of the conic to a pair of opposite $\left\{\begin{matrix}\text{sides}\\\text{vertices}\end{matrix}\right.$ has a constant ratio to the rectangle of its distances from the other pair.

7. If from any two points the vertices of a triangle be projected upon

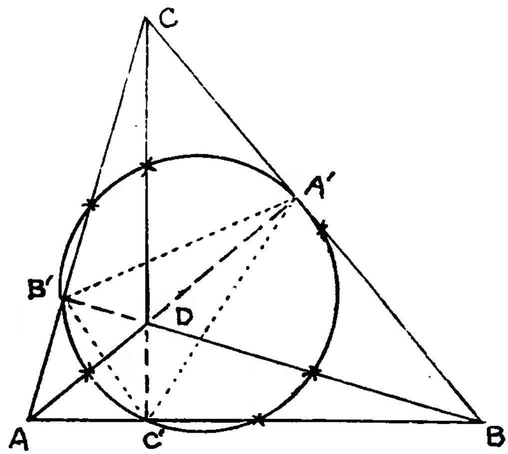
the opposite sides, the six points so found lie on a conic.

8. State and prove the correlative of Ex. 7.

9. In $\left\{\begin{matrix}\text{Ex. 7}\\ \text{Ex. 8}\end{matrix}\right.$ replace the words "two $\left\{\begin{matrix}\text{points}\\ \text{lines}\end{matrix}\right.$" by "conic" and prove the resulting theorem.

10. The mid-points of the sides of a four-point lie on a conic through the diagonal points. [th. 48 or th. 45 cr., th. 50.

11. The mid-points of the sides of a triangle and the feet of the perpendiculars from the opposite vertices lie on a conic.



12. The conic found in Ex. 11 is a circle through the three mid-points of the co-point of the three perpendiculars and the three vertices [Ex. 10.

This circle is the *nine-point circle* of the triangle.

13. The circle circumscribed about a triangle passes through the six mid-points of the centres of the four inscribed circles [Ex. 12, ths. 28-29.

14. Construct the conic, given four $\left\{\begin{matrix}\text{points}\\ \text{tangents}\end{matrix}\right.$ and the $\left\{\begin{matrix}\text{tangent thro}\\ \text{contacts of}\end{matrix}\right.$ one of them.

15. So, given three $\left\{\begin{matrix}\text{points}\\ \text{tangents}\end{matrix}\right.$ and the $\left\{\begin{matrix}\text{tangents thro}\\ \text{contacts of}\end{matrix}\right.$ two of them.

16. To draw a conic $\left\{\begin{matrix}\text{thro}\\ \text{touching}\end{matrix}\right.$ a given $\left\{\begin{matrix}\text{point}\\ \text{line}\end{matrix}\right.$ and the unknown

$\left\{\begin{array}{l}\text{co-points}\\ \text{co-tangents}\end{array}\right.$ of two given conics.

17. To draw a conic $\left\{\begin{array}{l}\text{thro}\\ \text{touching}\end{array}\right.$ the unknown $\left\{\begin{array}{l}\text{co-points}\\ \text{co-tangents}\end{array}\right.$ of two given conics and $\left\{\begin{array}{l}\text{touching}\\ \text{thro}\end{array}\right.$ a given $\left\{\begin{array}{l}\text{line.}\\ \text{point.}\end{array}\right.$ [two solutions or none.

18. To draw a conic $\left\{\begin{array}{l}\text{thro}\\ \text{touching}\end{array}\right.$ three given $\left\{\begin{array}{l}\text{points}\\ \text{lines}\end{array}\right.$ and having double contact with a given conic. [four solutions, two or none.

When do the four solutions become two solutions and an improper conic taken twice?

19. To draw a conic $\left\{\begin{array}{l}\text{thro}\\ \text{touching}\end{array}\right.$ two given $\left\{\begin{array}{l}\text{points}\\ \text{lines}\end{array}\right.$ and having double contact with a given conic such that the $\left\{\begin{array}{l}\text{co-line of contact passes thro}\\ \text{co-point of tangents lies upon}\end{array}\right.$ a given $\left\{\begin{array}{l}\text{point.}\\ \text{line.}\end{array}\right.$ [two solutions or none.

20. To draw a conic, given three $\left\{\begin{array}{l}\text{points}\\ \text{tangents}\end{array}\right.$ and the osculating circle $\left\{\begin{array}{l}\text{thro}\\ \text{touching}\end{array}\right.$ one of them.

21. Given two pairs of conjugate $\left\{\begin{array}{l}\text{points}\\ \text{rays}\end{array}\right.$ of an involution, a circle $\left\{\begin{array}{l}\text{thro}\\ \text{touching}\end{array}\right.$ one pair of them, and the $\left\{\begin{array}{l}\text{tangents to}\\ \text{points of}\end{array}\right.$ the circle $\left\{\begin{array}{l}\text{thro}\\ \text{upon}\end{array}\right.$ the other pair, to construct the double elements of the involution.

22. To trisect a given arc of a circle by a rectangular hyperbola through the centre and one end of the arc.

23. To construct an hyperbola and its asymptotes, given:
four points and the direction of one asymptote;
three points and the direction of both asymptotes.

24. To construct an hyperbola and one asymptote, given
two points, one asymptote, and the direction of the other.

25. To construct an hyperbola, given
one point and both asymptotes.

26. So, given tangents instead of points.

27. To construct a parabola and its direction to infinity, given
four $\begin{cases}\text{points.} & \text{[Two solutions or none.}\\ \text{tangents.} & \text{[One solution.}\end{cases}$

28. To construct a parabola, given
three $\begin{cases}\text{points,}\\ \text{tangents,}\end{cases}$ and its direction to infinity.

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§ 11. CONIC RANGES AND PENCILS.

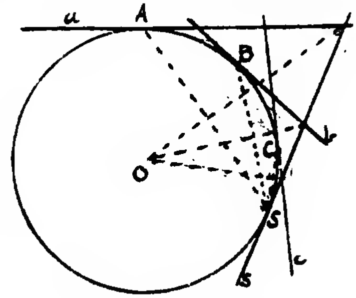
A group of $\begin{cases}\text{points upon}\\ \text{tangents to}\end{cases}$ a conic, taken in a prescribed order, is a conic $\begin{cases}\textit{range}\\ \textit{pencil}.\end{cases}$

In the light of theor. 47 the meaning of the word *projective* may be so extended that a conic $\begin{cases}\text{range}\\ \text{pencil}\end{cases}$ is projective with any $\begin{cases}\text{pencil based upon it}\\ \text{range cut from it}\end{cases}$ whose $\begin{cases}\text{centre lies upon}\\ \text{axis touches}\end{cases}$ the conic.

THEOR. 51. *A conic pencil is projective with the conic range formed by its points of contact.*

For, let the conic, the pencil, and its points of contact be the image

of a circle O, the pencil of tangents $a, b, c \cdots$, and their contacts $A, B, C \cdots$.



Cut the pencil $abc \cdots$ by any tangent s, and from S, the contact of s, project the range ABC; and from the centre O project the range s-$abc \cdots$;

then ∵ the rays O-sa, sb, $sc \cdots$ are perpendicular to the rays S-A, B, C$\cdots$, [Geom.

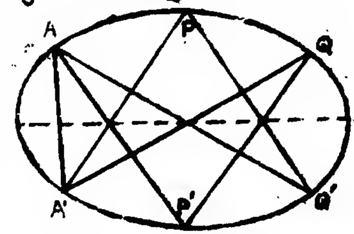
∴ these two pencils are directly equal;

∴ the range s-$abc \cdots$ and the pencil S-ABC$\cdots$ are projective,

∴ their images, the conic range and pencil, are projective. Q.E.D.

[THEOR. 52. *In two co-conic projective {ranges / pencils} the {co-line / co-point} of any pair of {points / lines} taken one from each {range / pencil} and the {co-line / co-point} of their images {reflect / project} into each other from a fixed {axis / centre.}*

For, let APQ$\cdots$, A′P′Q′$\cdots$ be any two co-conic projective ranges, wherein P, Q′ are any pair of points taken one from each range, and A is a fixed point;



then ∵ the pencils A-A′P′Q′$\cdots$, A′-APQ$\cdots$ are projective [hyp.

and the ray AA′ is its own image

∴ these pencils are in perspective with a fixed axis s [Th. 30, cor. 2.

and ∵ AP′, A′P, AQ′, A′Q are images and reflect into each other from axis s

∴ s is the Pascal line of the hexagon AP′QA′PQ′,

and PQ′, P′Q reflect into each other from s. Q.E.D. [Th 43

The fixed $\left\{\begin{matrix}\text{axis } s\\ \text{centre } S\end{matrix}\right.$ found above is the $\left\{\begin{matrix}\text{Pascal line}\\ \text{Brianchon point}\end{matrix}\right.$ of the two co-conic $\left\{\begin{matrix}\text{ranges}\\ \text{pencils.}\end{matrix}\right.$

The Brian. point of two projective conic pencils and the Pascal line of the projective ranges formed by their points of contact are *pole* and *polar* as to the conic.

COR. 1. *The double* $\left\{\begin{matrix}\textit{points}\\ \textit{rays}\end{matrix}\right.$ *of a pair of co-conic projective* $\left\{\begin{matrix}\textit{ranges}\\ \textit{pencils}\end{matrix}\right.$ *are the* $\left\{\begin{matrix}\textit{co-points}\\ \textit{co-tangents}\end{matrix}\right.$ *of the conic and the* $\left\{\begin{matrix}\textit{Pascal line;}\\ \textit{Brian. point;}\end{matrix}\right.$ *and, conversely.*

For, let Q be one of the co-points named, and Q' its image;
then $\because$ PQ', $P'Q$ reflect into each other from s and Q is on s, [th. hyp.
$\therefore$ PQ' passes through Q;
and $\because$ a line meets a conic in but two points, and $P \not\equiv Q$,
$\therefore$ $Q' \equiv Q$. Q.E.D.

Conversely, if $Q' \equiv Q$
then $\because$ PQ', $P'Q$ meet in Q,
$\therefore$ Q lies upon s and is a co-point of s and the conic. Q.E.D.

COR. 2. *The pair of tangents at the ends of a chord of a conic meet in the pole of that chord.*

For $\because$ the co-points of the Pascal line are double points of the two projective ranges,
$\therefore$ the tangents thereat are double rays of the reciprocal pencil and meet at the Brian. point. [th. 51. th. cr. 1.

COR. 3. *If a line* $\left\{\begin{matrix}\textit{lie without}\\ \textit{touch}\\ \textit{lie within}\end{matrix}\right.$ *a conic, its pole, as to that conic* $\left\{\begin{matrix}\textit{lies within}\\ \textit{is the contact of}\\ \textit{lies without}\end{matrix}\right.$ *the conic; and, conversely.*

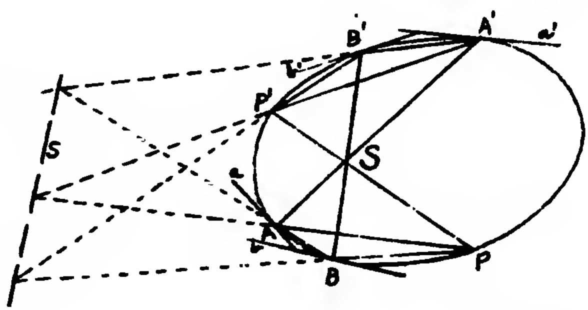
PROB. 13. *Given a circle, to find the double* {*points* / *rays*} *of two projective* {*coaxial ranges.* / *concentric pencils.*}

Project the ranges s-ABC··· s-A'B'C'··· from any point S of the circle, into the circular ranges LMN···, L'M'N'···. Let the co-line of LM'-L'M, LN'-L'N, (the Pascal line) meet the circle in H, K and take s-SH, s-SK for the required double points. [th.52.cr.1.

S.12. CONIC INVOLUTION.

If in two co-conic projective {ranges / pencils} every {point / tangent} has the same image to whichever figure it belongs the two {ranges / pencils} are in *conic involution.*

THEOR. 53. *In two co-conic* {*ranges* / *pencils*} *in involution every* {*point projects* / *tangent reflects*} *into its conjugate from the* {*Brian point;* / *Pascal line;*} *and, conversely, every* {*point* / *line*} *in the plane of the conic is the* {*Brian point* / *Pascal line*} *of an involution.*



For, let A, A', B, B', P, P' be pairs of conjugate points of a conic involution;

then ∵ AB, AP, BP reflect into A'B', A'P', B'P' from the Pascal line of the projective ranges ABB'P, A'B'BP', [th. 52.

∵ the triangles ABP, $A'B'P'$ are in perspective,

∴ AA', BB', PP' meet in a co-point S

Let aa', bb' be the pairs of rays of the reciprocal involution of tangents at A, A', B, B';

then ∵ $aa_1 \equiv A$, and $bb_1 \equiv B$, project into $a'a_1' \equiv A'$, and $b'b_1' \equiv B'$, from the Brian. point; [th. 52.

∴ $S \equiv AA' \text{-} BB'$, is the Brian. point. Q.E.D.

Conversely, let S be any point and through S draw two lines cutting the conic in A, A', B, B';

then the conic involution determined by the two pairs of points A, A', B, B' has S for Brian. point. Q.E.D.

COR. *In a conic involution with no double elements the arc between a pair of conjugates overlaps that between every other pair of conjugates; and, conversely.*

For, if there be no double elements the Brian. point S lies within the conic, [th. 52, cr. 1

and chords through it cut the conic in arcs that overlap. Q.E.D.

Conversely, if the arcs overlap, the chords meet within the conic, in the Brian. point, and there are no double elements. Q.E.D.

THEOR. 54. *Two co-conic involutions have one pair of conjugate elements in common unless each of them have two double elements such that the new involution determined by these two pairs is negative.*

For, let S, T be the Brian. points of two co-conic involutions;

then ∵ lines through S, T cut the conic in conjugate points of the two involutions [th. 53

∴ the line ST cuts the conic in a pair of points that are con-

jugate in both involutions;
and ∵ this line cuts the conic if either S or T lie within it,
∴ if either involution have no double point, the two involutions have in common a pair of conjugates. Q.E.D.

Let both involutions have double points H, K, M, N;
then ∵ the tangents h, k, m, n at H, K, M, N join pairs of conjugate points, viz: the two points of the double point H, those of K, of M, of N,
∴ $hk \equiv S$, $mn \equiv T$,
∴ ST is the Pascal line of the involution determined by the two pairs of points H, K, M, N; [df. Pas. line.
and ST cuts the conic unless the involution HK MN be negative.
Q.E.D. [th. 52. cr. 1.

COR. Two {coaxial / concentric} involutions have a common pair of conjugates unless the {segments / angles} between the double elements overlap.

NOTE. Theor. 43 may be proved by aid of this theorem as follows:

Let O be the co-centre of the two pencils and through O draw a conic cutting them in two involutions whose Brian. point is S, and with O as centre form another involution whose conjugate rays are at right angles;
then ∵ this last involution has no double ray,
∴ its rays cut the conic in an involution whose Brian. point lies within the conic. Let T be this point; and let the line ST cut the conic in H, K;
and OH, OK are conjugate rays of both involutions.

If the conic be a circle T is the centre. [Geom.

EXAMPLES.

1. Given a circle and an involution of $\left\{\begin{array}{l}\text{points,}\\\text{rays,}\end{array}\right.$ to construct the double elements.

2. Given a circle and two involutions of $\left\{\begin{array}{l}\text{points,}\\\text{rays,}\end{array}\right.$ to construct the common conjugates.

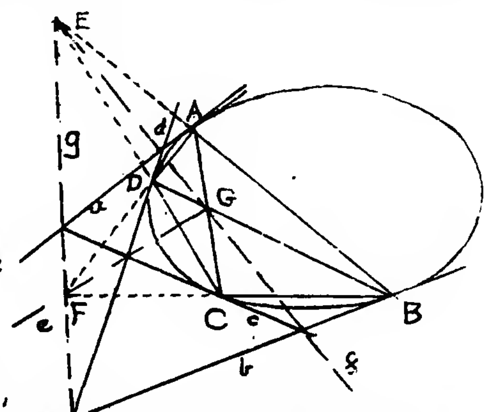
3. Given a circle and an involution of rays, to construct the conjugates that are at right angles.

In all these examples distinguish between the two cases:

(1) when the circle $\left\{\begin{array}{l}\text{touches}\\\text{passes thro}\end{array}\right.$ the $\left\{\begin{array}{l}\text{axis,}\\\text{centre,}\end{array}\right.$ (2) when it does not.

§ 13. CONJUGATE POINTS AND LINES.

THEOR. 55. *If at four points of a conic tangents be drawn, the four-point and the four-line so formed have the same diagonal triangle.*



For, let A, B, C, D be the four points, $a \equiv AA$, $b \equiv BB$, $c \equiv CC$, $d \equiv DD$, and let EFG be the diagonal triangle of ABCD and efg that of a.b.c.d,

then ∵ the hexagons AABCCD, ABBCDD have two pairs of common opposite sides AB, CD, BC, AD

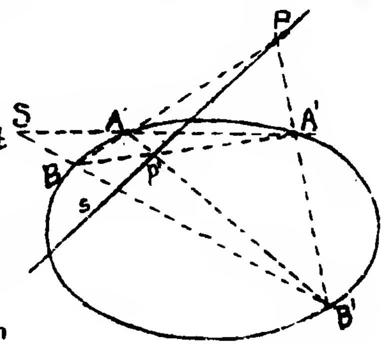
∴ they have the same Pascal line EF,

and ∵ the pairs of opposite sides AA, CC, BB, DD meet in the points ac, bd whose co-line, the Pascal line is g, th. 48

∴ $g \equiv EF$.

So $e \equiv FG$, $f \equiv GE$. Q.E.D.

<u>THEOR. 56.</u> *If of a variable* $\left\{\begin{array}{l}\textit{four-point inscribed in}\\ \textit{four-line circumscribed about}\end{array}\right.$ *a conic one diagonal* $\left\{\begin{array}{l}\textit{point}\\ \textit{line}\end{array}\right.$ *be fixed, the* $\left\{\begin{array}{l}\textit{locus}\\ \textit{envelop}\end{array}\right.$ *of the other two diagonal* $\left\{\begin{array}{l}\textit{points}\\ \textit{lines}\end{array}\right.$ *is the* $\left\{\begin{array}{l}\textit{polar}\\ \textit{pole}\end{array}\right.$ *of the fixed* $\left\{\begin{array}{l}\textit{point.}\\ \textit{line.}\end{array}\right.$



For, let $AA'BB'$ be a variable inscribed four-point such that $S, \equiv AA'-BB'$, is fixed;

then ∵ A, A', B, B' give the conic involution whose Brian. point is S,

∴ the diagonal points $AB'-A'B$, $AB-A'B'$ lie on the Pascal line, s, of the involution. Q.E.D.,

Conversely, let P be any point of the Pascal line, s; through S draw a chord AA'; let AP cut the conic again in B, and SB in B';

then ∵ A, A', B, B' are conjugate points of the involution whose Pascal line is s, [th. 53.

∴ $A'B'$ meets AB on s, and so at P; [th. 52.

∴ S, P are diagonal points of the four-point $AA'BB'$. Q.E.D.

COR. 1 *If two of the diagonal* $\left\{\begin{array}{l}\textit{points}\\ \textit{lines}\end{array}\right.$ *be fixed the third one is also fixed.*

COR. 2. *If a point move upon an axis, the polar of the point turns upon a centre, and axis and centre are polar and pole; and, conversely.*

<u>THEOR. 57.</u> *If a variable* $\left\{\begin{array}{l}\textit{four-point inscribed in}\\ \textit{four-line circumscribed about}\end{array}\right.$ *a conic so move that one* $\left\{\begin{array}{l}\textit{side}\\ \textit{vertex}\end{array}\right.$ *of its diagonal triangle be a fixed* $\left\{\begin{array}{l}\textit{line,}\\ \textit{point,}\end{array}\right.$ *the pairs of* $\left\{\begin{array}{l}\textit{vertices}\\ \textit{sides}\end{array}\right.$ *that* $\left\{\begin{array}{l}\textit{lie on}\\ \textit{pass thro}\end{array}\right.$ *that* $\left\{\begin{array}{l}\textit{line}\\ \textit{point}\end{array}\right.$ *are the conjugate* $\left\{\begin{array}{l}\textit{points}\\ \textit{lines}\end{array}\right.$ *of an involution.*

For, let s, [fig. th. 56], be a fixed line, P, P' a pair of variable diagonal points upon s; let S be the pole of s, and through S draw a chord AA', and construct the four-point $AA'BB'$ as in the converse of theor. 56, and having SPP' for its diagonal triangle;
then ∵ $AB, A'B$ generate two projective pencils, [th. 47].
∴ P, P' generate two projective ranges;
and ∵ the rays $AB', A'B'$ cut s in P', P,
∴ P has the same image to whichever range it belong,
and the two ranges are in involution. Q.E.D.

The diagonal $\left\{\begin{matrix}\text{points}\\ \text{lines}\end{matrix}\right.$ of $\left\{\begin{matrix}\text{an inscribed four-point}\\ \text{a circumscribed four-line}\end{matrix}\right.$ taken in pairs are *conjugate* as to the conic, and the involution of such conjugate $\left\{\begin{matrix}\text{points upon}\\ \text{lines thro}\end{matrix}\right.$ any $\left\{\begin{matrix}\text{line}\\ \text{point}\end{matrix}\right.$ is the involution *given* by the conic $\left\{\begin{matrix}\text{upon}\\ \text{thro}\end{matrix}\right.$ that $\left\{\begin{matrix}\text{line.}\\ \text{point.}\end{matrix}\right.$

COR. 1. *The* $\left\{\begin{matrix}\textit{co-points}\\ \textit{co tangents}\end{matrix}\right.$ *of the given* $\left\{\begin{matrix}\textit{line}\\ \textit{point}\end{matrix}\right.$ *and the conic, are the double* $\left\{\begin{matrix}\textit{points}\\ \textit{rays}\end{matrix}\right.$ *of the involution.*

For, let s cut the conic in H, K;
then ∵ the rays $AH, A'H$ cut s in the same point H,
∴ H is its own conjugate; and so for K. Q.E.D.

COR. 2. *A pair of* $\left\{\begin{matrix}\textit{points}\\ \textit{lines}\end{matrix}\right.$ *that are conjugate as to a conic are harmonic conjugates as to the* $\left\{\begin{matrix}\textit{co-points}\\ \textit{co-tangents}\end{matrix}\right.$ *of the conic with their* $\left\{\begin{matrix}\textit{co-line.}\\ \textit{co-point.}\end{matrix}\right.$

COR. 3. *If three* $\left\{\begin{array}{l}\textit{points}\\ \textit{lines}\end{array}\right.$ *be conjugate to each other as to a conic, two and two, so are their* $\left\{\begin{array}{l}\textit{co-lines.}\\ \textit{co-points.}\end{array}\right.$ [th. 54.

Such a triangle is a self conjugate triangle.

COR. 4. *In a self conjugate triangle two of the* $\left\{\begin{array}{l}\textit{vertices}\\ \textit{sides}\end{array}\right.$ *lie* $\left\{\begin{array}{l}\textit{without}\\ \textit{within}\end{array}\right.$ *the conic and one lies* $\left\{\begin{array}{l}\textit{within.}\\ \textit{without.}\end{array}\right.$

For ∵ if a vertex lie within the conic the locus of its conjugate cannot cut the conic, [th. 52, cr. 3.
∴ the other two vertices lie without;
and ∵ if a vertex lie without, the locus of its conjugates cuts the conic,
and the segment so formed is cut harmonically by any pair of conjugates, [cr. 2
∴ the other two vertices lie one within the conic and the other without.

Q.E.D.

COR. 5. *If a point trace a range its polar sweeps over a projective pencil; and, conversely.*

For, let P trace a range s - P ··· upon s;
then ∵ P′, the conjugate of P, traces a range s - P′ ··· projective with s - P ···,
∴ SP′, the polar of P, sweeps over a pencil projective with the range s - P ···.

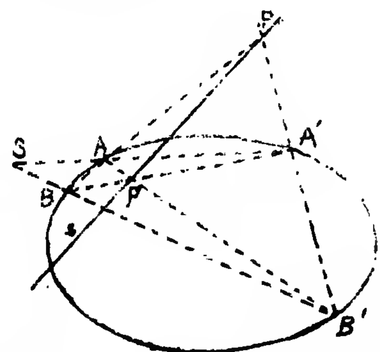
Q.E.D.

COR. 6. *A* $\left\{\begin{array}{l}\textit{range}\\ \textit{pencil}\end{array}\right.$ *is projective with any* $\left\{\begin{array}{l}\textit{range}\\ \textit{pencil}\end{array}\right.$ *of* $\left\{\begin{array}{l}\textit{points}\\ \textit{rays}\end{array}\right.$ *that are conjugate to the* $\left\{\begin{array}{l}\textit{points}\\ \textit{rays}\end{array}\right.$ *of the first* $\left\{\begin{array}{l}\textit{range.}\\ \textit{pencil.}\end{array}\right.$

For the second range is a section of the pencil of polars of the points of the first range.

COR. 7. *The* $\left\{\begin{array}{l}\text{co-points}\\ \text{co-lines}\end{array}\right.$ *of like* $\left\{\begin{array}{l}\text{rays}\\ \text{points}\end{array}\right.$ *of two* $\left\{\begin{array}{l}\text{pencils}\\ \text{ranges}\end{array}\right.$ *whose like* $\left\{\begin{array}{l}\text{rays}\\ \text{points}\end{array}\right.$ *are conjugate* $\left\{\begin{array}{l}\text{lines, trace}\\ \text{points, envelop}\end{array}\right.$ *a conic.*

THEOR. 58. *A given pair of* $\left\{\begin{array}{l}\text{points on}\\ \text{tangents to}\end{array}\right.$ *a conic,* $\left\{\begin{array}{l}\text{project}\\ \text{reflect}\end{array}\right.$ *from any* $\left\{\begin{array}{l}\text{point on}\\ \text{tangent to}\end{array}\right.$ *the conic* $\left\{\begin{array}{l}\text{upon}\\ \text{thro}\end{array}\right.$ *a conjugate of the* $\left\{\begin{array}{l}\text{co-line}\\ \text{co-point}\end{array}\right.$ *of the given pair into* $\left\{\begin{array}{l}\text{points}\\ \text{lines}\end{array}\right.$ *that are conjugate as to the conic.*



For, let B, B' be a given pair of points on a conic, A any other point on the conic, s a conjugate of BB', and on BB' take S the pole of s; and let SA cut the conic in A'.

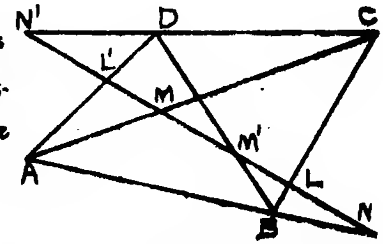
then ∵ AB, A'B', AB', A'B meet on s in the diagonal points P, P' of the four-point AA'BB',

∴ AB, AB' meet s in the conjugate points P, P'. Q.E.D.

COR. *The* $\left\{\begin{array}{l}\text{co-lines}\\ \text{co-points}\end{array}\right.$ *of a pair of* $\left\{\begin{array}{l}\text{points on}\\ \text{tangents to}\end{array}\right.$ *a conic and a pair of conjugate* $\left\{\begin{array}{l}\text{points on}\\ \text{lines thro}\end{array}\right.$ *a conjugate of the* $\left\{\begin{array}{l}\text{co-line}\\ \text{co-point}\end{array}\right.$ *of the first pair* $\left\{\begin{array}{l}\text{meet in points on}\\ \text{join by tangents to}\end{array}\right.$ *the conic.*

THEOR. 59. *If two pairs of opposite* $\left\{\begin{array}{l}\text{vertices}\\ \text{sides}\end{array}\right.$ *of a* $\left\{\begin{array}{l}\text{four-line}\\ \text{four-point}\end{array}\right.$ *be*

conjugate elements as to any conic in the plane of the {four-line, / four-point, the third pair are conjugate elements as to the conic.



For, let A, L, B, M, C, N be pairs of opposite vertices of a four-line formed by a triangle ABC and a transversal LMN; let A, L, B, M be two pairs of conjugate points, and D any convenient point, be the pole of LMN; let AD, BD, CD cut LMN in L'M'N';

then ∵ L, L', M, M', N, N' are pairs of conjugate points of the involution determined by the four-point ABCD. [th. 44.

and ∵ ADL', BDM' are the polars of L, M [th. 57

∴ L, L', M, M' are pairs of conjugate points of the involution determined by the conic.

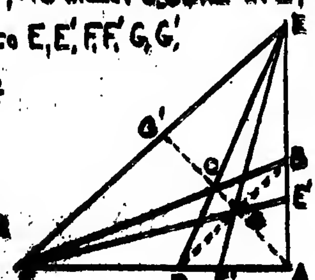
∴ N, N' are conjugates as to the conic,

and CDN' is the polar of N,

∴ C, N are conjugates as to the conic. Q.E.D.

THEOR. 60. *All conics that {pass thro / touch} a fixed {point / line} and have a single triangle self conjugate as to all of them {pass thro / touch} three other fixed {points. / lines.*

For, let A be the fixed point, EFG the triangle, let AE, AF, AG meet FG, GE, EF in E', F', G'; and construct B, D, C harmonic conjugates of A as to E, E', F, F', G, G',



then ∵ E is the pole of FG as to each one of the given conics,

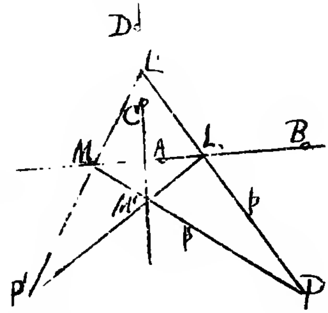
and ∵ B is the harmonic conjugate of A as to E E'

∴ B is a point of the conic. Q.E.D. [th. [illegible]

So D, C are points of the conic. Q.E.D.

COR. 1. *The two conics determined by four* $\left\{\begin{matrix}\text{points}\\ \text{tangents}\end{matrix}\right.$ *and one* $\left\{\begin{matrix}\text{tangent}\\ \text{point}\end{matrix}\right.$ *have three other* $\left\{\begin{matrix}\text{co-tangents}\\ \text{co-points,}\end{matrix}\right.$ *and a single self conjugate triangle.*

THEOR. 61. *Every* $\left\{\begin{matrix}\text{point}\\ \text{line}\end{matrix}\right.$ *in the plane of a* $\left\{\begin{matrix}\text{pencil}\\ \text{range}\end{matrix}\right.$ *of conics has a single conjugate as to all the conics of the* $\left\{\begin{matrix}\text{pencil;}\\ \text{range;}\end{matrix}\right.$ *and the pair seperate harmonically the opposite* $\left\{\begin{matrix}\text{sides}\\ \text{vertices}\end{matrix}\right.$ *of the* $\left\{\begin{matrix}\text{four-point}\\ \text{four-line}\end{matrix}\right.$ *of the* $\left\{\begin{matrix}\text{pencil}\\ \text{range}\end{matrix}\right.$ *of conics.*



For, let P be the given point and let A, B, C, D be the four points of the four-point of the pencil of conics, and so named that the angles APB, CPD do not overlap;

then ∵ the two involutions whose double rays are PA, PB, PC, PD have a pair of co-rays p, p' and cut AB, CD in involutions whose double points are A, B, C, D,

i.e. in the involutions on AB, CD given by every conic of the pencil;

∴ p, p' cut AB, CD in points L, L', M, M' that are conjugate as to every conic,

∴ $P' \equiv LM' \cdot L'M$, is conjugate to P as to every conic, [th. 59.

∴ PP' is cut harmonically by every conic of the pencil and in particular by the opposite sides of the four-point of the pencil. Q.E.D. [th. 57, cr. 2.

COR. 1. *The several* $\left\{\begin{matrix}\text{polars}\\ \text{poles}\end{matrix}\right.$ *of a fixed* $\left\{\begin{matrix}\text{point}\\ \text{line}\end{matrix}\right.$ *as to the conics of a given* $\left\{\begin{matrix}\text{pencil}\\ \text{range}\end{matrix}\right.$ *of conics* $\left\{\begin{matrix}\text{pass thro}\\ \text{lie on}\end{matrix}\right.$ *a fixed* $\left\{\begin{matrix}\text{point.}\\ \text{line.}\end{matrix}\right.$

COR. 2. *Two* $\left\{\begin{matrix}\text{points}\\ \text{lines}\end{matrix}\right.$ *that are conjugate as to each of two of the*

conics of a $\begin{cases}\text{pencil}\\\text{range}\end{cases}$ of conics are *conjugate* as to every conic of the $\begin{cases}\text{pencil.}\\\text{range.}\end{cases}$

THEOR. 62. *Two conics in perspective give the same involution upon the* $\begin{cases}\text{axis}\\\text{centre}\end{cases}$ *of perspective.*

For ∵ the diagonal points of like four-points are images,

∴ conjugate points as to one of the conics, on the axis, are conjugate points as to the like conic. Q.E.D.

COR. *If similar conics, similarly placed, pass through the same two points at infinity.*

The two points at infinity through which all circles pass are the *circular points at infinity.*

THEOR. 63. *If two conics give the same involution* $\begin{cases}\text{upon a line } s\\\text{at a point } S\end{cases}$ *either negative or positive, and such that a* $\begin{cases}\text{point}\\\text{ray}\end{cases}$ *of it lies* $\begin{cases}\text{within}\\\text{without}\end{cases}$ *both conics the two conics are in perspective, with* $\begin{cases}s \text{ as axis.}\\S \text{ as centre.}\end{cases}$

For, let G, G' be the poles of s as to the two conics, and let H be a point on s and, if the involution be positive, let H be a point within both conics,

then ∵ if the involution be negative G, G' lie within their conics,

and if the involution be positive, H lies within both conics,

∴ in either case, $GH, G'H$ cut their conics in real and seperate points A, B, A', B',

and ∵ s is conjugate to the chords $AB, A'B'$ as to their conics,

∴ the co-lines of $AB, A'B'$ and any pair of conjugate points E, F, on s meet in points P, Q, P', Q' of the conics, [th. 58. cr.

i.e. $AB, \cdots AP, AQ \cdots$ reflect from s into $A'B', \cdots A'P', A'Q', \cdots$

∴ the conics $AB\cdots PQ\cdots$, $A'B'\cdots P'Q'\cdots$ are in perspective with s as axis. [th. 6.

COR. 1. *If two conics pass through the same two imaginary points at infinity, or the same two real points at infinity so that a point at infinity lies within both conics, the two conics are similar and similarly placed.*

COR. 2. *All conics through the circular points at infinity are circles.*

The centers of perspective of two similar conics are *centers of similitude*; the axes of perspective are *radical axes*.

PROB. 14. *To construct the* $\left\{\begin{matrix}\textit{polar}\\ \textit{pole}\end{matrix}\right.$ *of a given* $\left\{\begin{matrix}\textit{point } S\\ \textit{axis } s\end{matrix}\right.$ *as to a given conic.*

1. By five $\left\{\begin{matrix}\text{points } A,B,C,D,E \text{ of}\\ \text{tangents } a,b,c,d,e \text{ to}\end{matrix}\right.$ the conic construct $\left\{\begin{matrix}A',B'\\ a',b'\end{matrix}\right.$, the $\left\{\begin{matrix}\text{co-points}\\ \text{co-tangents}\end{matrix}\right.$ of $\left\{\begin{matrix}SA, SB\\ sa, sb\end{matrix}\right.$ and the conic; the $\left\{\begin{matrix}\text{co-line } AB\text{-}A'B',\ AB'\text{-}A'B\\ \text{co-point } ab\text{-}a'b',\ ab'\text{-}a'b\end{matrix}\right.$ is the $\left\{\begin{matrix}\text{polar}\\ \text{pole}\end{matrix}\right.$ sought.

2. By five $\left\{\begin{matrix}\text{tangents } a,b,c,d,e \text{ to}\\ \text{points } A,B,C,D,E \text{ on}\end{matrix}\right.$ the conic and $\left\{\begin{matrix}\text{thro } S\\ \text{on } s\end{matrix}\right.$ construct two $\left\{\begin{matrix}\text{lines } p,q\\ \text{points } P,Q\end{matrix}\right.$ and their $\left\{\begin{matrix}\text{poles } P,Q;\\ \text{polars } p,q;\end{matrix}\right.$ the $\left\{\begin{matrix}\text{polar}\\ \text{pole}\end{matrix}\right.$ sought is $\left\{\begin{matrix}PQ.\\ pq.\end{matrix}\right.$

EXAMPLES.

1. $\left\{\begin{matrix}\text{Upon}\\ \text{At}\end{matrix}\right.$ a given $\left\{\begin{matrix}\text{line,}\\ \text{point,}\end{matrix}\right.$ to construct the involution given by a conic $\left\{\begin{matrix}A,B,C,D,E.\\ a,b,c,d,e.\end{matrix}\right.$

2. $\left\{\begin{matrix}\text{At}\\ \text{Upon}\end{matrix}\right.$ a given $\left\{\begin{matrix}\text{point,}\\ \text{line}\end{matrix}\right.$ to construct the involution given by a conic $\left\{\begin{matrix}A,B,C,D,E.\\ a,b,c,d,e.\end{matrix}\right.$

3. Given four $\left\{\begin{matrix}\text{points}\\ \text{tangents}\end{matrix}\right.$ and a pair of conjugate $\left\{\begin{matrix}\text{points,}\\ \text{lines,}\end{matrix}\right.$ to construct the conic.

4. So, given three $\left\{\begin{array}{l}\text{points}\\ \text{tangents}\end{array}\right.$ and an involution $\left\{\begin{array}{l}\text{upon an axis.}\\ \text{at a centre.}\end{array}\right.$

5. So, given two $\left\{\begin{array}{l}\text{points}\\ \text{tangents}\end{array}\right.$ and a self-conjugate triangle.

6. So, given one $\left\{\begin{array}{l}\text{point}\\ \text{tangent}\end{array}\right.$ and two $\left\{\begin{array}{l}\text{axial}\\ \text{central}\end{array}\right.$ involutions.

If one or both of the involutions be negative, find a conjugate of the given point by the method of theor. 61: it lies upon the tangent. Find a self-conjugate triangle, and find the involutions upon the two sides that pass through the co-point of the given axes; then on, as in theor. 58, &c.

7. To construct a conic $\left\{\begin{array}{l}\text{through}\\ \text{touching.}\end{array}\right.$ four $\left\{\begin{array}{l}\text{points}\\ \text{lines}\end{array}\right.$ and a pair of conjugate $\left\{\begin{array}{l}\text{points}\\ \text{lines}\end{array}\right.$ of a given involution.

8. A conic is in perspective with itself from any centre and axis that are pole and polar as to the conic; and the vanishing lines coincide midway between the centre and axis.

9. An arc of $\left\{\begin{array}{l}\text{an ellipse falls short of}\\ \text{a parabola touches}\\ \text{an hyperbola cuts}\end{array}\right.$ a line parallel to its chords and midway between that chord and the co-point of tangents at the ends of the chord.

10. If a $\left\{\begin{array}{l}\text{point slide}\\ \text{line turn}\end{array}\right.$ upon a fixed $\left\{\begin{array}{l}\text{line}\\ \text{point}\end{array}\right.$ the $\left\{\begin{array}{l}\text{locus}\\ \text{envelop}\end{array}\right.$ of its common conjugate as to two conics is a conic $\left\{\begin{array}{l}\text{thro}\\ \text{touching}\end{array}\right.$ three fixed $\left\{\begin{array}{l}\text{points,}\\ \text{lines,}\end{array}\right.$ whereof one is always real.

11. The three fixed $\left\{\begin{array}{l}\text{points}\\ \text{lines}\end{array}\right.$ found in ex. 7 are the $\left\{\begin{array}{l}\text{vertices}\\ \text{sides}\end{array}\right.$ of a triangle that is self conjugate as to each of the given conics; and if two of these

$\begin{cases}\text{points}\\ \text{lines}\end{cases}$ be imaginary, their $\begin{cases}\text{co-line}\\ \text{co-point}\end{cases}$ is real and the $\begin{cases}\text{segments cut therefrom}\\ \text{angles subtended thereat}\end{cases}$ by the given conics are real and overlap.

12. If two given conics have either four or no $\begin{cases}\text{co-points}\\ \text{co-tangents}\end{cases}$ their common self conjugate triangle is real; if they have but two $\begin{cases}\text{co-points}\\ \text{co-tangents}\end{cases}$ that triangle has but one real vertex and one (the opposite) real side.

13. If two conics be in homology they are also in homology as to a second axis and a second centre of homology; and they are in homology as to either of the two centers joined with either of the two axes; and the co-point of the two axes is the pole of the co-line of the two centers as to both the conics.

14. Any two co-planar conics are in homology, and there is either one pair of axes and centers, (as in Ex. 10) or three such pairs.

15. Two co-planar conics have four $\begin{cases}\text{co-points}\\ \text{co-tangents}\end{cases}$ (real or imaginary in pairs) and a pair of $\begin{cases}\text{axes}\\ \text{centers}\end{cases}$ (as in Ex. 10) are opposite $\begin{cases}\text{sides}\\ \text{vertices}\end{cases}$ of the $\begin{cases}\text{four-point}\\ \text{four-line}\end{cases}$ formed of these $\begin{cases}\text{co-points.}\\ \text{co-tangents.}\end{cases}$

16. Given two fixed points S, S', two fixed lines s, s' and two $\begin{cases}\text{segments } AB,\ A'B'\\ \text{angles } ab,\ a'b'\end{cases}$ of fixed $\begin{cases}\text{length}\\ \text{size}\end{cases}$ that $\begin{cases}\text{slide}\\ \text{turn}\end{cases}$ upon these $\begin{cases}\text{lines;}\\ \text{points;}\end{cases}$ to so place the two $\begin{cases}\text{segments}\\ \text{angles}\end{cases}$ that $\begin{cases}A, A', S\\ a, a', s\end{cases}$ are $\begin{cases}\text{colinear}\\ \text{concurrent}\end{cases}$ and so are $\begin{cases}B, B', S'.\\ b, b', s'.\end{cases}$

Slide AB, $A'B'$ on s, s' so that B, B', S' are colinear; then SA, SA' generate projective pencils, and either double ray is one of the positions sought for AA'.

17. Given two pairs of projective $\left\{\begin{array}{l}\text{ranges, upon}\\ \text{pencils, at}\end{array}\right.$ the $\left\{\begin{array}{l}\text{axes } s, s', t, t',\\ \text{centers } S, S', T, T',\end{array}\right.$ to find two $\left\{\begin{array}{l}\text{lines } a, a' \text{ thro}\\ \text{points } A, A' \text{ upon}\end{array}\right.$ a given $\left\{\begin{array}{l}\text{point } O\\ \text{line } o\end{array}\right.$ such that $\left\{\begin{array}{l}sa, s'a'\\ SA, S'A'\end{array}\right.$ are images, and so are $\left\{\begin{array}{l}ta, t'a'.\\ TA, T'A'.\end{array}\right.$

Through O draw a random line a' cutting s', t' in A', B', and find A, B the images upon s, t of A', B'; then, as a' varies, OA, OB generate projective pencils, and either double ray is one of the positions sought for a.

Discuss the special case when $\left\{\begin{array}{l}s, s'\\ S, S'\end{array}\right.$ coincide and so do $\left\{\begin{array}{l}t, t'.\\ T, T'.\end{array}\right.$

18. To construct a polygon that shall be circumscribed about a given n-point and inscribed in a given n-line.

Through the first of the n points draw a line at random to meet the first of the n lines; from the co-point so found draw a line through the second point to meet the second line, and so on till all of the n points have been passed and all of the n lines have been met. The co-point of the last line so drawn and the first will not in general be upon the last of the n lines; but these two lines generate projective pencils, and either of the double rays cuts the last side of the n-line in the vertex sought. Through this point and the first of the n points draw a line and retrace the whole figure.

19. Starting from a fixed point, draw a line to cut a fixed line,

from the co-point draw a second line to a second fixed point cutting a second fixed line and so on till n lines are drawn through n points cutting n lines, and so that the last line drawn makes a given angle with the first.

Discuss the special case wherein the first and second points are symmetric as to the first line, the second and third points as to the second line, and so on.

20. Through a given point to draw a line parallel to a given line.

21. From a given point to draw a line perpendicular to a given line.

22. To bisect a given segment of a line.

23. To bisect a given arc of a circle.

24. To bisect a given angle.

25. Given a segment AB, to produce it to C so that AC shall be the double of AB.

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§ 14. RECIPROCAL FIGURES.

Two figures lying in the plane of a conic, the one a point-figure, the other a line-figure that is made up of the polars, as to the conic, of the points of the first figure, are *reciprocal figures*. The conic is the *base*. A point and its polar are *reciprocal elements* of the two figures.

E.g., if a point trace a curve, the polar of the point envelops another curve, and the points of the first curve and the tangents to the other form two reciprocal figures.

THEOR. 64. *The co-line of two points of a point-figure and the co-point of the like two lines of the reciprocal figure are reciprocal elements.*

For, let $ABC\cdots$ be a point-figure and $abc\cdots$ the reciprocal line-figure taken to any conic as base;
then AB is the polar of ab, AC of ac, $\cdots$. Q.E.D. [th. 56, cr. 2

COR. *If two curves be so related that the points of the first are reciprocals to the tangents to the second then the points of the second are reciprocal to the tangents to the first.*

For the tangents to the first figure are co-lines of consecutive points and the reciprocal points of the second are the co-points of the reciprocal consecutive tangents. Q.E.D.

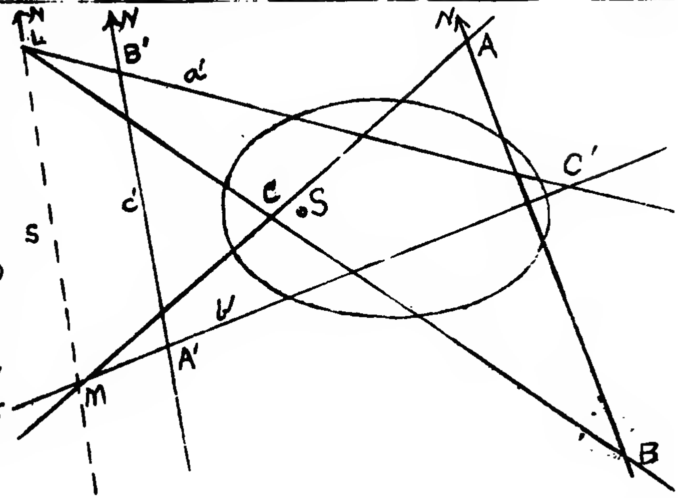
Two curves so related are *reciprocal curves*.

THEOR. 65. *The reciprocal of a conic is a conic.*

For, let p, p' be a pair of variable rays drawn from two fixed points of the given conic to a variable point pp' of the other, and let P, P' be the polars of p, p';
then the reciprocal curve is enveloped by the line PP'. [th. 64. df. recip. fig.
and $\because$ p, p' generate two projective pencils [th. 47
$\therefore$ P, P' generate two projective ranges [th. 57. cr. 5.
$\therefore$ the envelop of PP' is a conic. Q.E.D. [th. 47

COR. *If the pole of the line at infinity lie* $\left\{\begin{array}{l}\textit{within}\\ \textit{upon}\\ \textit{without}\end{array}\right.$ *a conic, the reciprocal of the conic is an* $\left\{\begin{array}{l}\textit{ellipse.}\\ \textit{parabola.}\\ \textit{hyperbola.}\end{array}\right.$

THEOR. 66. *If two triangles be reciprocal as to a conic they are in perspective.*



For, let ABC, A′B′C′ be two triangles such that ABC are poles of B′C′, C′A′, A′B′, and let L, M, N ≡ BC - B′C′, CA - C′A′, AB - LM;

then ∵ in the four-point formed by the triangle ABC and transversal LMN, L lies on B′C′ the polar of A, and M on C′A′ the polar of B,

∴ N lies on A′B′ the polar of C, [th. 59

∴ the triangles are in perspective Q. E. D. [th. 6

COR. *The centre and axis of this perspective are pole and polar.*

PROB. 15. *Given a pair of triangles reciprocal as to a conic, to construct the* $\left\{\begin{matrix}\textit{pole}\\ \textit{polar}\end{matrix}\right.$ *of a given* $\left\{\begin{matrix}\textit{line.}\\ \textit{point.}\end{matrix}\right.$

Let ABC, A′B′C′ be the pair of reciprocal triangles, s the axis of perspective, cutting BC, CA, AB in L, M, N, and S the centre. [th. 66

1. *P any point, to construct the pole of* AP:

Find P′ a point on C′B′L such that the range C′B′LP′ is projective with the pencil A-BCSP; P′ is the point sought. [th. 66, th. 57, cr. 5.

2. *P any point, to construct the polar of* P:

Construct the poles of AP, BP; their co-line is the line sought. [th. 66.

PROB. 16. *Given a pair of co-planar triangles in perspective, to construct the conic, if any, as to which the triangles are reciprocal.*

Let ABC, $A'B'C'$ be the given triangles, s the axis, S the centre; construct the system of poles and polars by prob. 15, and take the double points of the involution determined by two pairs of conjugate points on any line t, as points of the conic sought.

If a single positive involution may be so found, the conic is *real* [th. 60]; but if every such involution be negative, the conic is *imaginary*.

A system of points and lines that are poles and polars as to any conic, whether that conic be real or imaginary, is a *polar system*.

The method of reciprocal polars may be used to prove the correlatives of the projective properties of conics; for the reciprocal as to any conic of a figure of points, lines, and conics, is a correlative figure of lines, points, and conics, such that if a point lie upon a conic its reciprocal touches the reciprocal conic, and conversely, while the projective relations are unchanged.

EXAMPLES.

Construct the polar system, and the conic when real, given:

1. A self-conjugate triangle and a negative $\left\{\begin{array}{l}\text{axial}\\\text{radial}\end{array}\right.$ involution.

2. A self-conjugate triangle, a pole, and its polar.

3. Two negative $\left\{\begin{array}{l}\text{axial}\\\text{radial}\end{array}\right.$ involutions and a pair of conjugate $\left\{\begin{array}{l}\text{points.}\\\text{lines.}\end{array}\right.$

§ 15. CENTERS AND DIAMETERS.

The *centre* of a conic is the pole, as to that conic, of the line at infinity. An *axes* of a conic is a conjugate of the line at infinity, i.e. it is any line through the centre. Two axes that are conjugate rays of the radical involution given by a conic at its centre are *conjugate axes* of the conic. A *diameter* of a conic is a chord through the centre.

THEOR. 67. (a). *The centre of an* $\left\{\begin{array}{l}\textit{ellipse}\\ \textit{parabola}\\ \textit{hyperbola}\end{array}\right.$ *lies* $\left\{\begin{array}{l}\textit{within}\\ \textit{at infinity on}\\ \textit{without}\end{array}\right.$ *the curve.* [df. el. par. hyp. and th. 52 cr. 3.

(b). *The pairs of lines to infinity through the centre of a conic are the asymptotes of the conic, and they are the double rays of the involution of conjugate axes.*

(c). *The axes of a parabola are its* (parallel) *lines to infinity.*

(d). *The asymptotes of an hyperbola form a harmonic pencil with any pair of conjugate axes.*

(e). *In an* $\left\{\begin{array}{l}\textit{ellipse, each}\\ \textit{hyperbola, but one}\end{array}\right.$ *of a pair of conjugate axes cuts the curve.*

For the line at infinity forms a self-conjugate triangle with every pair of conjugate axes, of whose sides two only cut the conic. [th. 59. cr.

That axis of an hyperbola which cuts the curve is the *transverse axis.*

(f). *If a parallelogram circumscribe a conic, its diagonals are conjugate axes.*

For the line at infinity forms a self-conjugate triangle with the diagonals of this parallelogram.

THEOR. 68. *The lines that are conjugate, as to a conic, to a given axis of that conic, are parallel to the conjugate axis.*

For ∵ the pole of an axis lies on its conjugate, the line at infinity,
and ∵ all lines conjugate to the given axis pass through that pole, [th. 56 & conj. lines.
∴ these lines are parallel. Q.E.D.

THEOR. 69. *The locus of the mid-points of parallel chords of a conic is the axis conjugate to those chords.*

For, let HK be one of these chords, XX′ the axis conjugate to these chords, M the co-point of HK, XX′, and I the point at infinity on HK, i.e. the pole of XX′,
then ∵ I, M are conjugate points as to the conic [th. 56.
∴ M is the centre of the involution whose double points (or their ideals) are H, K,
i.e. M is the mid-point of HK Q.E.D.

COR. 1. *The centre of a conic is the mid-point of every diameter.*

COR. 2. *The ends of any two diameters are the vertices of a parallelogram; and, conversely, the diagonals of every parallelogram inscribed in a conic are diameters of the conic.*

COR. 3. *The tangents to a conic at the ends of a diameter are parallel to the conjugate diameter.*

COR. 4. *If from the ends of any diameter of a conic chords be drawn to any point of the conic, these chords are parallel to a pair of conjugate diameters; and, conversely.*

For the axis parallel to either chord bisects the other.

THEOR. 70. *A conic may have one and but one pair of right conjugate axes; or every pair may be at right angles, and the conic is then a circle.*

For, either the involution of conjugate axes has one and but one pair of rays at right angles.

or, if two pair be at right angles, and so, every pair;

then ∵ chords drawn from the ends of any diameter to any same point of the conic, are at right angles, [hyp. th. 69. cr. 4.

∴ the conic is a circle. Q.E.D.

In the parabola one of the right axes is the line at infinity and the other is the locus of the mid-points of chords perpendicular to the parallel axes of the parabola.

COR. 1. *The right axes of a conic are axes of symmetry; and, conversely, an axis of symmetry is one of the right axes.*

For, they bisect all chords perpendicular to them.

COR. 2. *The right axes of an hyperbola are the mid-lines of the asymptotes.*

COR. 3. *The segments of a line included between a conic and its asymptotes are equal.*

For, let the given line cut the conic in A, B, and the asymptotes in H, K

then ∵ the axis conjugate to that line passes through the centre M of the involution given by the double points A, B, [th. 69.

and through the centre of the involution given by the double points H, K, [th. 67 (b), th. 68.

∴ AM = MB, HM = MK,

∴ AH = KB, AK = HB. Q.E.D.

THEOR. 71. *A line in the plane of a conic is conjugate to the axis thro its pole, and its co-point with the axis is conjugate to its pole.*

For a line is conjugate to any line through its pole,
and the pole is conjugate to any point on its polar. [th. 56.

COR. 1. *The tangents at the ends of a chord meet on the axis that bisects the chord.*

The half chord is an *ordinate* to its conjugate axis, i.e. to the axis that bisects it, and the segment of the axis between the foot of an ordinate and the foot of the tangent at the end of the ordinate, is a *sub-tangent*.

COR. 2. *A subtangent is cut harmonically by the conic.*

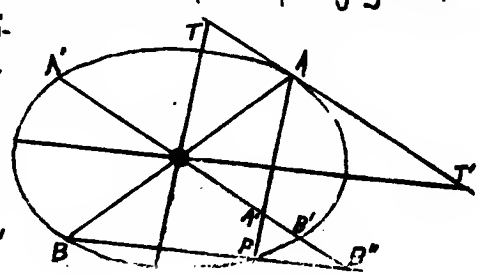
E.g., a parabola bisects its subtangent. See §13. Ex. 8.

COR. 3. *The angle between two lines is equal to the angle at the centre of a circle subtended by the poles, as to the circle, of the two lines.*

For, a line and the diameter through its pole are at right angles.

THEOR. 72. *A tangent to a conic cuts the radial involution of the conjugate axes of the conic in an axial involution whose power is the opposite of the power of the involution given by the conic upon the axis parallel to the tangent.*

For, let AB be any diameter of a conic and OT, OT′ a pair of conjugate axes cutting the tangent at A in T, T′, A′B′ the diameter parallel to TT′; from P a point of the conic such that AP is parallel to OT, project A, B into A″, B″ on A′B′;
then ∵ A′B′ is conjugate to AB, as to the conic,
∴ A is the centre of the involution T, T′



and $AT\cdot AT'$ is its power;

and ∵ A'', B'' are conjugate points as to the conic, and O is the centre, [th. 58.

∴ $OA''\cdot OB''$ is the power of the involution given by the conic on $A'B'$.

But ∵ AP is parallel to OT, [constr.

∴ $OTAA''$ is a parallelogram, and $AT = OA''$.

So ∵ O is the mid-point of AB, and BP is parallel to OT' [th. 69, cr. 4.

∴ triangles OAT', BOB'' are equal, and $AT' = OB''$

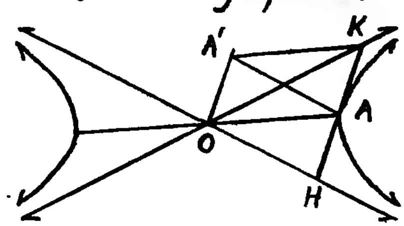
∴ $AT\cdot AT' = -OA''\cdot OB''$. Q.E.D.

COR. 1. *The product of the segments of a tangent to a conic between its contact and any pair of conjugate axes is constant and the opposite of the square of the half diameter parallel to the tangent.*

For, $OA''\cdot OB'' = \begin{cases}+\\-\end{cases} OB'^2$, the square of the $\begin{cases}\text{real}\\\text{imaginary}\end{cases}$ half diameter in the $\begin{cases}\text{ellipse.}\\\text{hyperbola.}\end{cases}$

COR. 2. *The segment of a tangent between the asymptotes is equal to the diameter parallel to the tangent; and its mid-point is its point of contact.* [th. 67 (b)

COR. 3. *In a parallelogram upon a pair of conjugate half diameters of an hyperbola, the diagonal through the the center is one asymptote, and the other diagonal is parallel to the other asymptote.*



For, let the tangent at A meet the asymptotes in H, K;

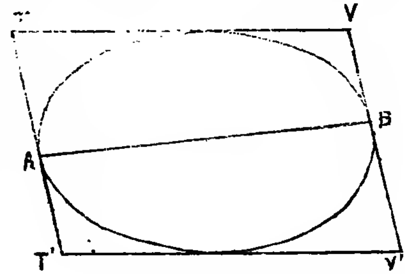
then ∵ $HA = AK$, and OA' is parallel and equal to them, [cr. 2.

∴ $OAKA'$, $OA'AH$ are parallelograms. Q.E.D.

COR. 4. *The product of the segments of a fixed tangent between its contact and a pair of variable parallel tangents is the opposite of the square of the half diameter parallel to the fixed tangent.*

For, the fixed tangent at A, its parallel tangent at B, and a pair of variable parallel tangents TV, T'V' form a circumscribing parallelogram TT'V'V whose diagonals VT', V'T are conjugate axes. [th. 67 (f).

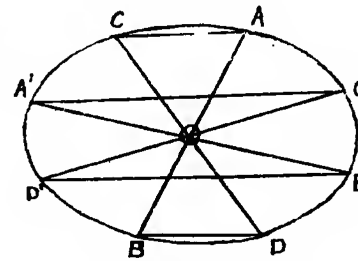
COR. 5. *The product of segments on fixed parallel tangents between their contacts and a variable tangent is equal to the square on the half diameter parallel to the fixed tangents.*



For $\because BV = -AT'$

$\therefore AT \cdot BV = -AT \cdot AT'$. Q.E.D.

THEOR. 73. *A pair of conjugate diameters of a conic project by parallel chords into any other pair of conjugate diameters of that conic.*



For, let AB, A'B' be any pair of conjugate diameters of the conic O, CD any other diameter; join AC, BD by parallel chords [th. 69, cr. 2], and draw A'C', B'D' parallel to AC, BD; draw OX conjugate to the parallel chords and OX' parallel to them;

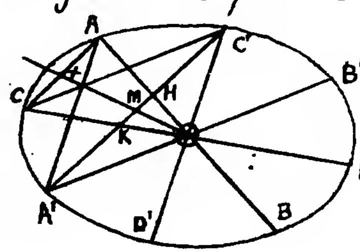
then $\because$ O-XACX', O-X'A'C'X are harmonic pencils, [OX, OX' bisecting and parallel to AC, A'C'.

$\therefore$ OX, OX', OA, OA', OC, OC' are pairs of conjugate rays in involution;

and $\because$ the first two pairs are conjugate axes, [hyp.

$\therefore$ OC, OC' are conjugate axes Q.E.D. [df. conj. axes.

THEOR. 74. *The parallelogram on two conjugate half diameters of a conic has a constant area.*



For, let AB, A'B', CD, C'D' be any two pairs of conjugate diameters of the conic O, and let OA, OC, meet A'C' in H, K;

then $\because$ AC, A'C' are parallel, [th. 73.

∴ their mid-points X, M, are colinear with O, and $HM = MK$,

∴ Δs $OHA' = OC'K$, $A'HA = KC'C$, [eq. bases and alts.

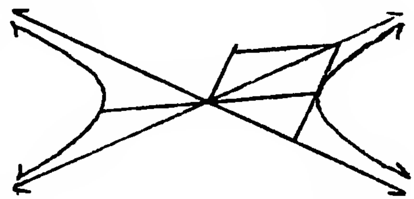
∴ Δs $OAA' = OCC'$, and so for their doubles. Q.E.D.

COR. 1. *The parallelogram on two half diameters of a conic is equal to the parallelogram on their conjugates.*

For ∵ Δs OKA', $A'KC = OC'H$, $HC'A$ [above.

∴ Δs $OA'C = OC'A$, and so for their doubles. Q.E.D.

COR. 2. *The triangle of the asymptotes and a tangent has a constant area equal to the parallelogram on a pair of conjugate half diameters.*



THE RIGHT HYPERBOLA. [§10, Ex. 3.

THEOR. 75. *The co-points of like rays of opposite pencils lie on an hyperbola whose asymptotes are at right angles and the two centers are at the two ends of a diameter; and, conversely, any hyperbola with right asymptotes may be so generated.*

COR. 1. *The asymptotes of a right hyperbola bisect the angles between any pair of conjugate axes.*

COR. 2. *The angles between any two axes of a right hyperbola are equal to the angles between their conjugates.*

COR. 3. *The circle circumscribing any self-conjugate triangle of a right hyperbola passes through the centre of the hyperbola.*

COR. 4. *The locus of the centre of a right hyperbola circumscribing a given triangle is its nine point circle.*

THEOR. 76. *Any pair of conjugate diameters of a right hyperbola are equal in length; and, conversely, if a single pair of conjugate diameters of an hyperbola be equal in length, the hyperbola is a right hyperbola.*

THEOR. 77. *If a conic circumscribe a triangle and pass thro its orthocentre* (co-point of perpendiculars from vertices to opposite sides) *it is a right hyperbola; and, conversely, a right hyperbola thro three points, passes thro the orthocenter of their triangle.*

For ∵ the opposite sides of the inscribed four-point, ABCD, formed by the vertices of the triangle and its orthocentre are right conjugate lines,

∴ they determine on the line at infinity, the involution given by a radial involution of right conjugate lines. be 56

∴ the asymptotes of the circumscribed conic are at right angles. [th. 49.

2. Thro ABC pass any right hyperbola cutting the line at infinity in J, K,

then ∵ the conic ABCDJ is a right hyperbola [th.

∴ it passes thro K, [OJ, OK at rt. angles

∴ ABCJK coincides with ABCDJK. Q.E.D. [th. 47. cr. 2

COR. *If a right triangle be inscribed in a right hyperbola the tangent at the vertex of the right angle is perpendicular to the hypothenuse.*

PROB. 17. *To find the centre and a pair of conjugate diameters of a conic.*

(a) *Given five points of the conic A, B, C, D, E:*

Through A, B, draw parallels to BC, CA meeting the conic in D', D'', then the diagonal triangles of the four-points $ABCD'$, $ABCD''$ have the vertices $AD'\text{-}BC$, $BD''\text{-}AC$ at infinity, and the opposite sides GE, $E'F'$ are axes of the conic, their co-point O is the centre, and their conjugates are parallel to AD', BD''.

Take OA' the opposite of OA; then AA' is a diameter.

Construct EE' the conjugate of AA'.

Through A, A' draw parallels to a pair of conjugate axes, meeting upon the conic and cutting EE' in points P, P' that are conjugate as to the conic; then OE the half diameter conjugate to OA is equal to $\sqrt{OP \cdot OP'}$.

To find the right axes, in the involution of conjugate axes, construct the pair of conjugate rays that are at right angles.

Or, through A, B, C draw a circle, and find its fourth co-point F, with the conic.

Through O draw parallels to the mid-lines of a pair of opposite sides of the four-point $ABCF$; they are the axes sought:

For ∵ the points at infinity on the right axes of a conic are conjugate as to the conic,

and ∵ they are also conjugate as to the circle ABC, [th. 70.

∴ they lie on the bisectors of the angles named. Q.E.F. [th. 61, th. 28.

(b.) *Given five tangents to the conic, a, b, c, d, e:*

Parallel to a, b, c, draw tangents a', b', c' and take the diagonals of the parallelograms $bcb'c'$, $cac'a'$, $aba'b'$, as conjugate axes.

The axis conjugate to a, a' cuts them at A, A' the ends of a diameter.

From this point on the work is as in case (a).

The centre alone is found as the co-point of the bisectors of the diagonals of any two four-lines formed from the five given tangents. [th. 61. cr. 1.

PROB. 18. *Given a transverse half diameter OA and its conjugate OB, to construct pairs of conjugate half diameters of an* $\left\{\begin{array}{l}\textit{ellipse.}\\ \textit{hyperbola.}\end{array}\right.$

At A draw a parallel and a perpendicular to OB, being the tangent and normal to the conic at A; and on the normal take points C, D such that $OC = -OD = OB$;

then any circle that $\left\{\begin{array}{l}\text{passes thro C, D}\\ \text{cuts at right angles the circle on CD as diameter}\end{array}\right.$

cuts the tangents in points T, T′ such that $AT \cdot AT' = \mp OB^2$, and OT, OT′ are conjugate axes.

From A draw ordinates to these axes meeting them in V, V′ and take OE, OF such that $OE = \sqrt{OT \cdot OV}$, $OF = \sqrt{OT' \cdot OV'}$; then OE, OF are the conjugate half diameters sought. [th. 71, cr. 2.

The circle $\left\{\begin{array}{l}OCD\\ OO'O''\end{array}\right.$ gives the right axes, wherein OO′ is bisected at right angles by the tangent and OO″ is a segment on OA that is cut harmonically by the circle on CD as diameter. [th. 27, cr. 2.

PROB. 19. *Given a pair of conjugate diameters of a conic to construct* $\left\{\begin{array}{l}\textit{points of}\\ \textit{tangents to}\end{array}\right.$ *the conic.*

Let AA′ be a transverse diameter, BB′ its conjugate, a, a′ the tangents at A, A′.

Through any pair of conjugate points on BB′ draw lines $\left\{\begin{array}{l}\text{through A, A′;}\\ \text{parallel to AA′;}\end{array}\right.$ these lines $\left\{\begin{array}{l}\text{meet in points of}\\ \text{meet a, a′ upon tangents to}\end{array}\right.$ the conic. [th. 58, cr.

EXAMPLES.

1. Given two $\left\{\begin{array}{l}\text{points of}\\ \text{tangents to}\end{array}\right.$ a conic and a pair of conjugate axes, to construct the conic.

2. So, given a tangent, its point of contact, and a pair of conjugate axes.

3. So, given a $\left\{\begin{array}{l}\text{point}\\ \text{tangent}\end{array}\right.$ and two pairs of conjugate axes.

4. In the data above, let a tangent be the line at infinity.

5. To construct a right hyperbola $\left\{\begin{array}{l}\text{through}\\ \text{touching}\end{array}\right.$ four $\left\{\begin{array}{l}\text{points.}\\ \text{lines.}\end{array}\right.$

6. To construct a pair of conjugate axes of a conic whose angle is given.
Solve by co-points with a circle through the ends of a transverse diameter.

7. If a pair of conjugate diameters of an $\left\{\begin{array}{l}\text{ellipse}\\ \text{hyperbola}\end{array}\right.$ be projected upon any fixed diameter by ordinates to it, the $\left\{\begin{array}{l}\text{sum}\\ \text{difference}\end{array}\right.$ of the squares on the projections equals the square on the fixed diameter.

8. In the $\left\{\begin{array}{l}\text{ellipse}\\ \text{hyperbola}\end{array}\right.$ the $\left\{\begin{array}{l}\text{sum}\\ \text{difference}\end{array}\right.$ of the squares on a pair of conjugate diameters is constant.

9. If from a variable point P, lines be drawn to cut a conic in two fixed directions, the product of the segments between the point and the conic taken in one direction has a constant ratio to the product of the segment taken in the other direction. [Lth. Sa

10. If from a variable point on an asymptote to an hyperbola lines be drawn in a fixed direction the product of the two segments between the point and the hyperbola is constant.

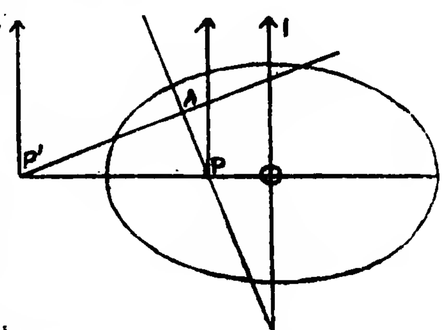
11. If an hyperbola and an asymptote be cut by a transversal in points A, A′, H, the product AH·HA′ equals the square of the parallel half-diameter.

12. If an hyperbola and its asymptotes be cut by a transversal in points A, H, H′, the product HA·AH′ equals the square of the parallel half-diameter.

13. From ex. 12, find a construction for the right diameters of an hyperbola, given a pair of conjugate half-diameters and the asymptotes.

§16. FOCI AND DIRECTRICES.

THEOR. 78. *A pair of right lines that are conjugate as to a conic cut either of the right axes in points such that every other pair of right lines thro them are conjugate as to the conic; and pairs of points so found are pairs of conjugate points of an involution upon the axis.*



1. Let AP, AP′ be a pair of right lines conjugate as to the conic O, P, P′ their copoints with one of the right axes, and I (at infinity) the pole of this axis;

then ∵ PA, P′A, PI, P′P, PP′, P′I are three pairs of right conjugate lines thro P, P′ [hyp.

and ∵ the pencils of conjugate rays P-AIP′ …, P′-API … are projective, [th. 59. cr. 6

∴ the two pencils are equal, and every ray thro P meets at right angles its conjugate thro P′. Q.E.D.

2. Let P, P′ Q, Q′ be pairs of such points upon the axis; thro P, P′, Q, Q′ … draw a pencil of parallel rays J-PP′QQ′ …, and another such pencil K-PP′Q′Q … whose rays are at right angles to the rays of the first;

then ∵ $J-PP'QQ'\cdots$, $K-P'PQ'Q\cdots$ are projective pencils, [above, th. 57 cor.

∴ P,P' $Q,Q'\cdots$ are pairs of conjugate points in involution. Q.E.D.

The double points of these involutions are the *foci* of the conic, the right axes are *focal axes*, and the involutions upon them are *focal involutions*.

COR. 1. *The centre of a focal involution is at the centre of the conic; and, of the two focal involutions, one is positive and the other negative.*

For either of the right axes and a parallel to the other are right conjugate lines, and they cut the other axis at the centre of the conic and at infinity;

and every pair of right lines cuts two other right lines, one on the same side of their co-point, and the other on opposite sides.

That focal axis upon which lie the real foci is the *major axis*; the other is the *minor axis*. The polars of the foci are the *directrices*.

COR. 2. *The real directrices are lines at right angles to the major axis and equally distant from the centre of the conic.*

For ∵ the polars of the real foci, F, F' are conjugates of the major axis,

∴ they are perpendicular to that axis,

and ∵ the feet, D, D' of these polars are conjugates of F, F',

∴ $OF\cdot OD = OA^2 = OF'\cdot OD'$, wherein OA is the half major diameter, [th. 38.

∴ $DO = OD'$ Q.E.D. [cr. 1

COR. 3. *The point and line at infinity of a parabola are focus and directrix, the centre and minor axis, a point and the tangent thereat, all in one.*

THEOR. 79. *At a focus and at no other point, a conic gives a radial involution of right conjugate lines.*

For, every pair of right lines through a focus are conjugate as to the conic, [th. 78. df. focus

and conversely if F be a point such that every pair of right lines thro

it are conjugate lines,

then the conjugate points of a focal involution subtend a right angle at F, [th. 78.

i.e. every circle thro a pair of conjugate points as diametric points pass thro F [geom.

and ∵ the foci of the other focal involutions are the only such points, [above.

∴ F is one of the foci. Q.E.D.

COR. 1. *The foci of a conic lie within the curve and are the copoints of the pairs of (imaginary) tangents to the conic from the circular points at infinity.*

COR. 2. *The major axis of a conic cuts the conic in real points.*

These points are the *vertices* of the conic.

COR. 3. *A pair of right conjugate lines (e.g. a tangent and normal) cut the major axis harmonically as to the foci, and they cut the minor axis in the diameter of the circle through their copoint and the foci.*

The focus of a parabola bisects the segment on the major axis between a pair of right conjugate lines.

COR. 4. *A tangent to a conic cuts the tangents at its vertices in the diametric points P, P′ of a circle through the foci; and P, P′ as vertices and F, F′ as diagonal points, determine a four-point whose side opposite the tangent PP′ is the normal.*

For ∵ the colines of P, P′ with any point on the major axis are conjugate lines [th. 85.

∴ PFP′, PF′P′ are right angles; [th. 78.

and ∵ in the given four-point two pairs of sides are right conjugate lines,

∴ the third pair are right conjugate lines; [geom. th. 59.

and ∵ one of these lines is a tangent to the conic,

∴ the other is a normal. Q.E.D.

COR. 5. *A pair of right conjugate lines are the mid-lines of the focal radii of their co-point and of the tangents from their co-point.*

COR. 6. *The focal radius of the co-point of two tangents to a conic and the focal radius of the co-point of their chord of contact with the directrix are the mid-lines of the focal radii to the points of contact.*

COR. 7. *The segments of a tangent between its contact and a directrix subtend a right angle at the focus.*

THEOR. 80. *The locus of the foci of parabolas that touch three fixed lines is a circle through the three co-points of the lines.*

For, let F be the focus, and I the point at infinity of a parabola touching the lines a, b, c whose co-points are C, A, B;

then ∵ ∠s BAC, FAI have the same mid-lines [th. 79. cr. 5.

∴ ∠s c-AI = ∠s FAC,

so ∠s c-BI = ∠s FBC,

∴ ∠ FAC = ∠ FBC,

and F lies on the circle ABC. Q.E.D.

COR. *If a parabola touch four fixed lines, its focus is the co-point of the four circles that pass thro the co-points of the lines, taken three and three.*

THEOR. 81. *The segments of a tangent, between two fixed tangents to a conic subtend at a focus one half the angle subtended by the chord of contact of the two tangents; and, conversely, a segment between fixed lines that subtends a constant angle at a fixed point envelops a conic that touches the fixed lines and has the fixed point for a focus.*

1. For, let tangents at A, B of a conic meet in T and cut a tangent at P in M, N; and let F be a focus;

then ∵ $\angle MFP = \frac{1}{2}\angle AFC$, $\angle PFN = \frac{1}{2}\angle PFB$ [th. 79. cr. 3.

∴ $\angle MFN = \frac{1}{2}\angle AFB$. Q.E.D.

2. ∵ the segment of every tangent to the conic determined by the fixed lines and one position of MN, as tangents, and a right radial involution at F will be a position of MN, [th. hyp.

and ∵ the pencils $F\text{-}M\cdots$, $F\text{-}N\cdots$ are equal [hyp.

∴ $M\cdots$, $N\cdots$ are projective ranges

and MN touches the conic in every position. Q.E.D. [th. 47.

THEOR. 82. *The power of a focal involution is the power of the involution given by the conic on the axis less the power of the involution given by the conic on the other focal axis.*

For, let O be a conic and T a point upon it, let the tangent and normal at T cut the given focal axis at P, P' and the other focal axis in Q, Q', and let P_1, Q_1 be the feet of the ordinates on the axes from T;

then ∵ $OP : OQ = Q_1T : Q_1Q = P_1T : P'P_1 = OQ_1 : P'O + OP_1$, [sim. triangles.

∴ $OP \cdot OP' = OP \cdot OP_1 - OQ \cdot OQ_1$. Q.E.D

COR. 1. *The square of the segment of the major axis between the center and a focus is the difference of the squares of the half major and the half minor diameters.*

COR. 2. *The half minor diameter is the geometric mean of the focal segments of the major diameter.*

For ∵ $AF \cdot FA' = (AO + OF)(AO - OF) = OA^2 - OF^2$ th. 78, cr. 1

and ∵ $OF^2 = OA^2 \mp OB^2$ [cor. 1

∴ $AF \cdot FA' = \pm OB^2$ Q.E.D.

The ratio of the segment of the major axis between the foci, to the major diameter, is the *eccentricity* of the conic.

COR. 3. *The eccentricity of a circle is 0, of an ellipse is less than 1, and of a parabola is 1, of an hyperbola greater than 1.*

COR. 4. *The ratio of the distances of a vertex of a conic from a focus and from its directrix equals the eccentricity of the conic.*

For $\because OF \cdot OD = OA^2$

$\therefore e = OF : OA = OA : OD = \underline{AO + OF : DO + OA} = AF : DA$. Q.E.D.

THEOR. 83. *The ratio of the distances of a point of a conic from a focus and from its directrix equals the eccentricity of the conic; and, conversely, if a point so move that the ratio of its distances from a fixed point and from a fixed line is constant, the point traces a conic, with the fixed point and line as focus and directrix.*

1. For, let $A \cdots P$ be a conic, F, f a focus and directrix, $A' \cdots P'$ an image of $A \cdots P$ from F as centre and with f as vanishing line;

then $\because$ F and the line at infinity are pole and polar as to $A' \cdots P'$, [image of F, f.

$\therefore F$ is the center of $A' \cdots P'$;

and $\because$ the radial involution at F is the same for either conic, [th. 62

$\therefore A' \cdots P'$ is a circle, and FP' is constant.

But $\because FP : fP = FP' : c$, wherein c is some constant, [th. 23, cor.

$\therefore FP : fP = c = FA : fA = e$. Q.E.D. [th. 82, cor. 4.

2. $\because FP : fP = c$, [hyp. conv.

$\therefore FP' = c$, and $A \cdots P$ is the image of a circle $A' \cdots P'$ with F as center;

and $\because$ F is the center of perspective and f the vanishing line,

$\therefore F, f$ are pole and polar as to $A \cdots P$, [image of pole and polar

and the radial involution at $\underline{F \text{ is a right}}$ involution. Q.E.D.

THEOR. 84. *The $\begin{cases}\text{sum}\\ \text{difference}\end{cases}$ of the focal distances of a point on an $\begin{cases}\text{ellipse}\\ \text{hyperbola}\end{cases}$ is constant and equal to the major diameter; and, conversely, if a point so move that the $\begin{cases}\text{sum}\\ \text{difference}\end{cases}$ of its distances from two fixed points is*

constant, it traces an $\left\{\begin{matrix}\textit{ellipse}\\ \textit{hyperbola}\end{matrix}\right.$ *with the fixed points as foci.*

1\. For, let F, F' be the foci of a conic, PQ, PQ' the directrix distances of a point P on the conic;

then $\because FP : PQ = F'P : PQ' = e$ [th. 83.

$\therefore FP \pm F'P = PQ \pm PQ' = e;$

and $\because PQ \pm PQ' = QQ'$, a constant,

$\therefore FP \pm F'P$ is constant

$= FA \pm F'A =$ major diameter. Q.E.D.

2\. For, every point of the $\left\{\begin{matrix}\text{ellipse}\\ \text{hyperbola}\end{matrix}\right.$ through one position of P and with F, F' as foci, is a position of P [th.,

and the focal distances of a point P' without or within this conic have a $\left\{\begin{matrix}\text{greater or less sum}\\ \text{less or greater difference}\end{matrix}\right.$ than the given point. [Geom.

COR. 1. *The major diameter of an* $\left\{\begin{matrix}\textit{ellipse}\\ \textit{hyperbola}\end{matrix}\right.$ *is its* $\left\{\begin{matrix}\textit{longest}\\ \textit{shortest}\end{matrix}\right.$ *diameter; and the minor diameter of an ellipse is its shortest diameter.*

COR. 2. *Two confocal conics have no real common tangent, and they cut each other at right angles in four real points* (if one conic be an ellipse and the other an hyperbola) *or in none.*

Confocal parabolas cut each other at right angles in two finite points, or in none.

A circle cuts every pair of lines through its centre at right angles.

THEOR. 85. *The locus of the foot of the perpendicular from a focus of a conic to a variable tangent is a circle whose diameter is the major diameter of the conic.*

For, let O be the centre of a conic, AA′ its major diameter, F, F′ the foci, P a point on the conic, PQ a tangent, FQ the perpendicular from F to PQ, R the co-point of FQ, F′P;

then ∵ PO is the mid-line of PF, PR and is perpendicular to FQ, [th. 79, cr., hyp.

$\therefore FQ = QR,\ FP = PR$

and ∵ $FO = OF'$

$\therefore OQ = \frac{1}{2}F'R = \frac{1}{2}(F'P \pm PR) = \frac{1}{2}AA'$ Q.E.D. [th. 84.

COR. 1. *The product of the distances from a focus to a pair of parallel tangents is constant.*

COR. 2. *The product of the distances of the foci from a tangent is the square of the half minor diameter; and, conversely, if a line so move that the product of its distances from two fixed points is constant, it envelops a conic with the fixed points as foci.*

The circle on the major diameter of a conic is the *major contact circle*. In the parabola this circle breaks up into the tangents at the vertices.

THEOR. 86. *The locus of the co-point of a pair of right tangents to a conic is a circle concentric with the conic with radius whose square is the sum of the squares of the half-major and half-minor diameters.*

For, let a pair of right tangents p, q thro T cut the major contact circle in P, P′; Q, Q′;

then ∵ $TP \cdot TP' = OT^2 - OA^2$ [geom.

and ∵ $TP = pF,\ TP' = pF'$ [th. 85, geom.

$\therefore TP \cdot TP' = pF \cdot pF' = \pm OB^2$ [th. 85, cr. 2.

$\therefore OT^2 = OA^2 \pm OB^2$ Q.E.D.

This circle is the *director-circle* of the conic.

COR. *The director-circle of a parabola breaks up into the directrix and the line at infinity.*

The director circle of a right hyperbola reduces to a point, the center; and if in a hyperbola $OB^2 > OA^2$, the director circle is imaginary.

PROB. 20. *Given the two foci and a tangent, to construct the conic:*

Take O, the mid-point of FF', as center and foot of the perpendicular from either focus to the tangent as a point of the major contact circle, and find A, A' the vertices of the conic, and a, a' the tangents at A, A'; through F, F' draw any circle cutting a, a' in diametric points B, B'; BB' is a tangent to the conic; and the harmonic conjugate as to F, F' of the foot of BB' is a point of the normal.

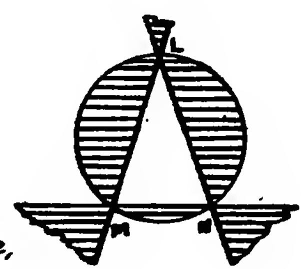
The normal may also be found as in theor. 79, cr. 4.

PROB. 21. *Given a focus and three tangents, to construct the conic:*

Let F be the given focus and L, M, N be the co-points of the tangents. Between LM, LN construct a segment M'N' such that $\angle M'FN' = \angle MFN$; then M'N' is tangent to the conic.

Take F' such that $\angle LMF' = \angle FMN$, $\angle LNF' = \angle FNM$; F' is the second focus

EXAMPLES.



1. Show that in prob. 21, the conic is an $\left\{\begin{array}{l}\text{ellipse}\\ \text{parabola}\\ \text{hyperbola}\end{array}\right.$ if F lie on the $\left\{\begin{array}{l}\text{white space}\\ \text{circle LMN}\\ \text{shaded space}\end{array}\right.$ of the figure.

2. Given two $\left\{\begin{array}{l}\text{points}\\ \text{tangents}\end{array}\right.$ and the major contact circle, to construct the conic.

3. If a sphere be inscribed in a cone of revolution and touch a plane, the contact of the sphere and plane is a focus of the section of the cone by the plane, and the co-line of this plane with the plane of the circle of con-

tact of the cone and sphere, is a directrix.

4. In ex. 3, if the inclination of the cutting plane to the axis of the cone be $\left\{\begin{array}{l}\text{greater than}\\ \text{equal to}\\ \text{less than}\end{array}\right.$ the half vertical angle of the cone, the section is $\left\{\begin{array}{l}\text{an ellipse.}\\ \text{a parabola.}\\ \text{an hyperbola.}\end{array}\right.$

5. An ellipse is the orthogonal projection of a circle turned through an angle whose cosine is the ratio of the minor to the major diameter.